A pyramid-shaped blow-up surface for the 2d semilinear wave equation

Hatem ZAAG

CNRS and LAGA Université Paris 13

Nonlinear Waves 2016 : the June Conference

IHES, Bures-sur-Yvette, June 20 to 24, 2016

Joint work with:
F. Merle (Université de Cergy-Pontoise and IHES).
Introduction: The equation

\[
\begin{cases}
\partial_t^2 u = \Delta u + |u|^{p-1}u, \\
  u(0) = u_0 \text{ and } u_t(0) = u_1,
\end{cases}
\]

where \( 1 < p < p_c = 1 + \frac{4}{N-1} \), \( u(t) : x \in \mathbb{R}^N \rightarrow u(x, t) \in \mathbb{R} \), \( u_0 \in H^1(\mathbb{R}^N) \) and \( u_1 \in L^2(\mathbb{R}^N) \).

Rk.: The conformal exponent \( p_c = 1 + \frac{4}{N-1} < 1 + \frac{4}{N-2} \), the Sobolev exponent.

Singular solutions: the maximal influence domain

We consider an arbitrary blow-up solution $u(x, t)$.
From the finite speed of propagation, its domain of definition is

$$D_u = \{(x, t) \mid 0 \leq t < T(x)\}$$

where $x \mapsto T(x)$ is 1-Lipschitz.

**Remark**: For all $x \in \mathbb{R}^N$, there exists a “local” blow-up time $T(x)$. 
Definition: Non characteristic points and characteristic points

A point $a$ is said non characteristic if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.

The point is said characteristic if not.

- Notation: $\mathcal{R} \subset \mathbb{R}^N$ is the set of all non characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}^N$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}^N$).
Case $N = 1$ (and $p > 1$): Existence results

**Rk.** All blow-up solutions have non-characteristic points ($x_0 = \text{arg min } T(x)$);

**Th (Merle, Z.):** There exist solutions with characteristic points.

**Example:** We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0), u(0, t) = 0$.

**Th. (Merle-Z.)** If we perturb the constructed initial data, then the new solution blows up and has a characteristic point.
Case $N = 1$ (and $p > 1$): Asymptotic behavior

Introducing similarity variables

$$w_{x_0}(y, s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x, t) \text{ with } y = \frac{x - x_0}{T(x_0) - t} \quad \text{and} \quad s = -\log(T(x_0) - t),$$

and the soliton

$$\kappa(d, y) = \kappa_0(p) \left(1 - d^2\right)^{\frac{1}{p-1}} (1 + dy)^{-\frac{1}{p-1}},$$

we have as $s \to \infty$:

- if $x_0 \in \mathbb{R}$, then $w_{x_0}(y, s) \to \pm \kappa(d(x_0), y)$;
- if $x_0 \in \mathbb{S}$, then $w_{x_0}(y, s) \sim \pm \sum_{i=1}^{k} (-1)^i \kappa(d_i(s), y)$ (multi-solitons)

with

$$k \geq 2 \text{ and } d_i(s) = \tanh(C_0(i - \frac{k+1}{2}) \log s + C_1).$$

Th. (Côte, Z.) : Every multi-soliton modality does occur.
Illustration with hyperbolic coordinates when $x_0 \in S$

Introducing for $\xi \in \mathbb{R}$,

$$w_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s)$$

with $y = \tanh \xi$ and $\zeta_i(s) = C_0(i - \frac{k + 1}{2}) \log s + C_1$,

we get

$$\|w_{x_0}(\xi, s) - \epsilon(x_0) \kappa_0 \sum_{i=1}^{k(x_0)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0$$
as $s \to \infty$,

and with $k(x_0) = 4$ and $\epsilon(x_0) = -1$:
Behavior of the solitons’ centers

\((\zeta_i)_{i=1,...,k}\) is a solution to the system

\[
\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}, \quad i = 1, \ldots, k,
\]

with the convention \(\zeta_0(s) \equiv -\infty, \zeta_{k+1}(s) \equiv +\infty\). Note that the barycenter is conserved

\[
\frac{1}{k}(\zeta_1(s) + \cdots + \zeta_k(s)) \equiv \bar{\zeta}(x_0).
\]

One can compute explicitly:

\[
\zeta_i(s) = \left(i - \frac{k+1}{2}\right)\frac{(p-1)}{2} \ln s + \alpha_i + \bar{\zeta}(x_0),
\]

where \(\alpha_i = \alpha_i(p, k)\) are chosen adequately.
Regularity of the blow-up curve

- $\mathcal{R}$ is open and $T|_\mathcal{R}$ is $C^1$; more precisely, if $d(x_0)$ is such that $w_{x_0}(y, s) \sim \pm \kappa(d(x_0), y)$, then $T'(x_0) = d(x_0)$;
- $S$ is finite on compact sets, and $T$ is corner shaped near $a \in S$.

Furthermore, for some $\gamma = \gamma(p) > 0$,

$$T(x) - T(x_0) + |x - x_0| \sim \frac{\gamma e^{2\zeta_0 \text{sgn}(x_0 - x)}|x - x_0|}{\ln |x - x_0|} \frac{1}{(k(x_0) - 1)(p - 1)} \text{ as } x \to x_0,$$

where $k(x_0)$ is the solitons’ number and $\zeta_0(x_0)$ is their barycenter. Note that
- The number of solitons $k(x_0)$ can be “seen” on the blow-up curve.
- The blow-up curve is never symmetric with respect to $x_0$, unless maybe if the barycenter of the solitons $\zeta_0(x_0) = 0$. 

Generalizations

- **When* $N = 1$ with lower order perturbations (M.A. Hamza and Z. 2013):**

  \[ \partial_t^2 u = \partial_x^2 u + |u|^{p-1} u + f(u) + g(\partial_t u, \partial_x u, x, t) \]

  with

  \[ |f(u)| \leq C(1 + |u|^q), \quad |g(\partial_t u, \partial_x u, x, t)| \leq M(1 + |\partial_t u| + |\partial_x u|) \]

  and $q < p$.

- **When* $N \geq 2$, $p < p_c$, with radial symmetry, outside the origin:**

  \[ \partial_t^2 u = \partial_r^2 u + (N - 1) \frac{\partial_r u}{r} + |u|^{p-1} u \]

  (this is because the term $\frac{\partial_r u}{r}$ appears as a lower order perturbation).

- A mixture of both cases (radial + perturbations),

- **When* $u \in \mathbb{C}$ (by A. Azaiez 2013).

- **With “strong” perturbations (by M.A. Hamza and O. Saidi), i.e.**

  \[ |f(u)| \leq C|u|^p \log^{-a} |u| \]
And what about $N \geq 2$ with $u$ not necessarily radial?

We know the blow-up rate (Merle, Z. 2003 and 2005):

- If $x_0 \in \mathcal{R}$, then
  \[
  0 < \epsilon_0(N,p) \leq \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<1)} \leq K(u_0, u_1);
  \]

- If $x_0 \in \mathcal{S}$, then
  \[
  \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(|y|<\frac{1}{2})} \leq K(u_0, u_1).
  \]

Facts about characteristic points when $N \geq 2$:
- No classification (except for radial solutions outside the origin);
- The only known examples are rigorously radial or 1d in some neighborhood of the characteristic point.

Question: Can we find “new” blow-up solutions with characteristic points, i.e. with a “non 1d”-behavior?
General question: Geometry of the set of singular points

Our dream: Find a solution where $\mathcal{S}$ is cross-shaped (open question)?

More generally, the geometry of the singular set is largely open in PDEs.

For the semilinear heat equation (Sobolev subcritical): We have solutions where the singular set is a single points, a finite number of points, a sphere, a finite number of concentric spheres. That’s it.

An ellipse in 2d? open problem.
A cross in 2d? open problem.

For the semilinear wave equation: The good notion for singular points concerns characteristic points, since all points are blow-up points, though at different times.
A new blow-up solution when $N = 2$

Th (Merle, Z. 2016) There exists a solution $u(x, t)$ which blows up on a 1-Lipschitz graph $x \mapsto T(x)$ such that

$$T(x) - T(0) \sim - \max(|x_1|, |x_2|) \text{ as } x \to 0 \text{ (pyramid shape)}$$

with the origin being an isolated characteristic point.

Rk.
- $u(x, t)$ is of course not radial.
- $u(x, t)$ is symmetric with respect to the axis and antisymmetric with respect to bisectrices. In particular $u(x_1, \pm x_1, t) = 0$.
- We could prove that the origin is the only characteristic point.
Regularity of the blow-up graph

Locally near 0, the blow-up graph

\[ x \mapsto T(x) \]

has the the regularity of the asymptotic pyramid

\[ x \mapsto T(0) - \max(|x_1|, |x_2|). \]

In particular:
- It is \( C^1 \) outside the bisectrices with (when \( 0 \leq x_2 < x_1 \))
  
  \[ \partial_{x_1} T(x) = -1 + c_0(p) |\log x_1|^{-\frac{p-1}{2}} + \ldots \text{ and } \partial_{x_2} T(x) = O(|\log x_1|^{-\frac{p-1}{4}}) \text{ as } x \to 0; \]

- On the bisectrices outside the origin, \( x \mapsto T(x) \) has directional derivatives except in the direction of the bisectrix;
- At the origin, we have directional derivatives, except along the bisectrices.

**Rk.** Unlike the 1d case, we have on the bisectrices the first example of non characteristic points where \( T \) is non differentiable.
A new solution in 2d

The blow-up behavior of $u(x, t)$ at the origin

If $x_0 = 0$, then (decoupled multi-solitons localized along the axes)

$$\left\| w_0(y, s) - (\kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y)) \right\|_{\mathcal{H}} \to 0$$

where

$$\kappa(d, y) = \kappa_0(p) \frac{(1 - d^2)^{\frac{1}{p-1}}}{(1 + dy)^{\frac{2}{p-1}}},$$

$$\bar{d}(s) = -\tanh \bar{\zeta}(s) \text{ and } \bar{\zeta}(s) = \left(\frac{p - 1}{4}\right) \log s - \frac{1}{4} \log \left(\frac{p - 1}{4c_4(p)}\right)$$

which is an explicit solution to the ODE

$$\frac{1}{c_4(p)} \bar{\zeta}'(s) = e^{-\frac{4}{p-1} \bar{\zeta}(s)}.$$

Rk. Note that $\bar{d}(s) = -1 + c_5(p)s^{-\frac{p-1}{2}} + \ldots$ as $s \to \infty$. 

Hatem ZAAG (P13 & CNRS)  Blow-up for NLW  Nonlinear Waves 2016 : the June Conference
The blow-up behavior of $u(x, t)$ outside the origin

If $x_0 \neq 0$, then $w_{x_0}(s) \to w^*_{x_0}$, a stationary solution in similarity variables with:

- $w^*_{x_0} = \kappa(d(x_0))$ if $0 \leq x_{0,2} < x_{0,1}$, and

$$d(x_0) = -1 + c_0 |\log x_{0,1}|^{-\frac{p-1}{2}} + \ldots \text{ as } x_0 \to 0,$$

for some $c_0(p) > 0$, with similar behavior whenever $|x_{0,1}| \neq |x_{0,2}|$, from symmetry;

- $w^*_{x_0}$ is a genuinely two-dimensional stationary solution if $|x_{0,1}| = |x_{0,2}|$.

Rk.
- $w^*_{x_0}$ is non radial; it is a new stationary solution.
The proof

Two major steps:

- **Step 1**: The construction in the light cone with vertex \((0, T(0))\) (i.e. the construction of \(w_0\)).
- **Step 2**: Derivation of the behavior of \(w_x(s)\) as \(s \to \infty\) and the behavior of \(x \mapsto T(x)\), for \(x\) small.

Rk.
- Between the two steps, using the finite speed of propagation, we extend the solution to the region outside the cone, in order to get a solution to the Cauchy problem at \(t = 0\).
- As usual with blow-up problems (heat, wave), the asymptotic behavior of the solution at blow-up and the regularity of the blow-up set are linked and advanced side by side in the proof.
Step 1: The construction in the central light cone

**Goal**: To construct a solution in similarity variables $w_0(y, s)$ for $|y| < 1$ and $s \geq s_0$ showing 4 solitons:

$$w_0(y, s) \sim \kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y) \quad \text{as} \quad s \to \infty,$$

where

$$\kappa(d, y) = \kappa_0(p) \frac{(1 - |d|^2)^{\frac{1}{p-1}}}{(1 + d \cdot y)^{\frac{2}{p-1}}},$$

(here, $d \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$),

$$\bar{d}(s) = -\tanh \bar{\zeta}(s) \quad \text{and} \quad \bar{\zeta}(s) = \left(\frac{p-1}{4}\right) \log s - \frac{(p-1)}{4} \log \left(\frac{p-1}{4c_4(p)}\right).$$
Step 1: The construction in the central light cone

**Framework**: Construction of a solution with *prescribed behavior*.  
**Method**: We linearize the equation around the intended behavior, and find three regions in the spectrum:  
- **Negative spectrum**: controlled thanks to a linearized version of the Lyapunov functional;  
- \( \lambda = 0 \): controlled thanks to modulation in the parameter \( d \) in \( \kappa(d, y) \);  
- \( \lambda = 1 \): controlled thanks to modulation in the parameter \( \nu \) in the *generalized solitons*:

\[
\kappa^*(d, \nu, y) = \kappa_0 \frac{(1 - |d|^2)^{\frac{1}{p-1}}}{(1 + \nu + d \cdot y)^{\frac{2}{p-1}}}.
\]

**Rk.** We were inspired by the construction of multi-solitons in 1d in Côte-Z. (2013).
History of the construction with prescribed behavior

More generally, we are in the framework of constructing a solution to some PDE with some \textit{prescribed behavior}:

- NLS: Merle (1990), Martel and Merle (2006); Côte, Martel and Merle (2011)
- KdV (and gKdV): Martel (2005), Côte (2006, 2007), Côte, Martel and Merle,
- water waves: Ming-Rousset-Tzvetkov (2013),
- Schrödinger maps: Merle-Raphaël-Rodniansky (2013),
- semilinear wave equation: Côte and Z. (2013),
- semilinear heat equation: Bressan, Merle (1992), Bricmont and Kupiainen (1993), Merle and Z. (1997), Shweyer,
- Keller-Segel: Raphaël-Schweyer, Ghoul-Masmoudi,
- etc.
Step 2: Behavior of $w_{x_0}$ and $T(x_0)$ for $x_0 \neq 0$

Take $x_0 \neq 0$.

We need to know the behavior of $w_{x_0}$ for $|y| < 1$ and $s \geq -\log T(x_0)$.

This is equivalent to knowing the behavior of $u(x, t)$ in the backward light cone $C_{x_0}$ with vertex $(x_0, T(x_0))$.

But, if $x_0$ is small, and $t < \min(T(0), T(x_0))$, the sections of $C_{x_0}$ and $C_0$ are almost the same.

Moreover, we have the following relation between $w_{x_0}$ and $w_0$:

$$w_{x_0}(y, s) = (1 - T(x_0)e^s)^{-\frac{2}{p-1}}w_0(Y, S), \quad Y = \frac{y + xe^s}{1 - T(x_0)e^s}, \quad S = s - \log(1 - T(x_0)e^s).$$

Since $w_0$ shows 4 solitons:

$$w_0(y, s) \sim \kappa(\tilde{d}(s)e_1, y) + \kappa(-\tilde{d}(s)e_1, y) - \kappa(\tilde{d}(s)e_2, y) - \kappa(-\tilde{d}(s)e_2, y) \text{ as } s \to \infty,$$

the function $w_{x_0}$ also shows 4 (generalized) solitons, though with a deformation.
Step 2: Behavior of $w_{x_0}$ and $T(x_0)$ outside the bisectrises

Two cases then arise:

**Case 1**: If $x_0$ is not on the bisectrices (say, $0 \leq x_{0,2} < x_{0,1}$), only one soliton remains at some time $t^* = T(x_0) - e^{-s^*} = T(0) - e^{-S^*}$:

$$w_{x_0}(y, s^*) \sim \kappa(d(S^*)e_1) \text{ with } S^* \sim -\log x_1.$$

Applying our trapping result near solitons, we see that if $x_0$ is non characteristic, then

$$w_{x_0}(y, s) \to \kappa(\nabla T(x_0), y) \text{ as } s \to \infty$$

with

$$\nabla T(x_0) \sim d(S^*)e_1 = (-1 + c_0S^* - \frac{p-1}{2} + ...)e_1 = (-1 + c_0|\log x_1|^{-\frac{p-1}{2}} + ...)e_1.$$

**Rk.**
- If $x_0$ is characteristic, we have no information; later, we will have to show that all points outside the bisectrices are non characteristic.
- Note the link between the asymptotic behavior of $w_{x_0}$ and the regularity of $T(x_0)$ in (1).
Step 2: Behavior of $w_{x_0}$ and $T(x_0)$ on the bisectrices

**Case 2:** If $x_0$ is on the bisectrices (say, $x_{0,2} = x_{0,1}$), then $w_{x_0}$ is anti-symmetric with respect to the bisectrix; therefore, 2 solitons remain at some time $\tilde{t} = T(x_0) - e^{-\tilde{s}} = T(0) - e^{-\tilde{S}}$:

$$w_{x_0}(y, \tilde{s}) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2)$$

with

$$\tilde{S} \sim - \log x_1.$$

From the behavior of the neighbors outside the bisectrix, we derive that $x_0$ is non-characteristic. Therefore, from the existence of a Lyapunov functional in similarity variables, we see that

$$w_{x_0}(y, s) \to w^*_{x_0}(y),$$

a stationary solution in similarity variables, with

$$w^*_{x_0}(y) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2).$$

**Rk.** This is a *new* kind of stationary solutions, which are neither radial, nor 1d.
Step 2: Outside the bisectrices, all the points are non characteristic (The Umbrella Technique)

**Goal:** Take \( x \) outside the bisectrices, for example with \( 0 \leq x_2 < x_1 \), and show that \( x \) is non characteristic.

**Proof:** Take \( \gamma \in (0, x_1^2] \) and consider a family of cones with vertex \((x, t)\) with \( t \leq T(x) \) and slope \( 1 - \gamma < 1 \).

Consider \( \bar{t} \) the largest value such of \( t \) such that the cone touches the graph of \( x \mapsto T(x) \) at some point \((\bar{x}, T(\bar{x}))\) (imagine an umbrella under the graph).

Since the slope of the cone is \( 1 - \gamma < 1 \), by definition, \( \bar{x} \) is non characteristic.

If \( \bar{x} = x \) (the graph touches the umbrella at its vertex), then \( x \) is non characteristic; we are done.
The Umbrella Technique (cont.)

If \( \bar{x} \neq x \), we will reach a contradiction.

If \( \bar{x} \) is on the bisectrices, this is a bit complicated to explain: omitted.

If \( \bar{x} \) is not on the bisectrices, say, \( 0 \leq \bar{x}_2 < \bar{x}_1 \), then both the cone and the graph are differentiable, and their slopes have to agree:

- Slope of the cone: \( -1 + \gamma \leq -1 + x_1^2 \),
- Slope of the graph: \( -1 + c_0 |\log \bar{x}_1|^{\frac{p-1}{2}} + \ldots \)

Therefore,

\[
c_0 |\log \bar{x}_1|^{\frac{p-1}{2}} + \ldots = \gamma \leq x_1^2, \text{ hence } \bar{x}_1 \ll x_1,
\]

on the one hand.
The Umbrella Technique (cont.)

On the other hand, since \((x, T(x))\) is the vertex of the umbrella and \((\bar{x}, T(\bar{x}))\) is on the umbrella, it follows that

\[ T(x) \geq T(\bar{x}). \]  \(2\)

Since \(x \mapsto T(x)\) is 1-Lipschitz and \(\bar{x}_1 \geq \bar{x}_2\), it follows that

\[ T(\bar{x}) \geq T(0) - |\bar{x}| \geq T(0) - \bar{x}_1 \sqrt{2}. \]  \(3\)

Since “\(w_x\) is bounded” by our work in 2003-2005, we do have the following (non sharp) upper bound:

\[ T(0) - \frac{x_1}{2} \geq T(x). \]  \(4\)

Combining (2), (3) and (4), we see that

\[ T(0) - \frac{x_1}{2} \geq T(x) \geq T(\bar{x}) \geq T(0) - \bar{x}_1 \sqrt{2}, \text{ hence } x_1 \leq 2\bar{x}_1 \sqrt{2}. \]

Recalling from the previous slide that \(c_0|\log \bar{x}_1|^{-\frac{p-1}{2}} + \ldots = \gamma \leq x_1^2\), hence \(\bar{x}_1 \ll x_1\),

we get a contradiction.
Thus, \textit{all points x outside the bisectrices are non characteristic} and

\[
\nabla T(x) = e_1(-1 + c_0|\log x_1|^{-\frac{p-1}{2}}) + \ldots \text{ if } 0 \leq x_2 < x_1.
\]

Integrating this estimate between 0 and \(x\) gives

\[
T(x) - T(0) \sim -x_1 \text{ if } 0 \leq x_2 < x_1.
\]

Extending this by symmetry, we obtain the \textit{pyramid shape}:

\[
T(x) - T(0) \sim -\max(|x_1|, |x_2|).
\]
Thank you for your attention