A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up

Tokyo University, December 17, 2009

Hatem ZAAG
LAGA, CNRS UMR 7539
Université Paris 13

joint work with Nejla Nouaili
Université Paris Dauphine

December 17, 2009
The equation

\[
\begin{align*}
\partial_t u &= \Delta u + (1 + i\delta)|u|^{p-1}u \\
u(0, x) &= u_0(x) \in L^\infty(\mathbb{R}^N),
\end{align*}
\]

(Equation \(\text{Equ}_\delta\))

where \(u(t) : \mathbb{R}^N \rightarrow \mathbb{C}, p > 1\) and \(\delta \in \mathbb{R}\).

We say that \(u(t)\) blows up in finite time \(T\), if \(u(t)\) exists for all \(t \in [0, T]\) and \(\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty\).

The point \(a\) is a blow-up point if and only if there exists \((a_n, t_n) \rightarrow (a, T)\) as \(n \rightarrow +\infty\) such that \(|u(a_n, t_n)| \rightarrow +\infty\).
Why this equation?

- A submodel of the Ginzburg-Landau equation

\[ \partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u - \gamma u \quad (1) \]

where \( \beta, \delta \) and \( \gamma \) are real (See Masmoudi and Zaag JFA 2008 where a blow-up solution is contructed for equation (1)).

- A lab model for the blow-up problem in parabolic equations with no gradient structure.
Outline of the talk

1. Case $\delta = 0$, $(N - 2)p < N + 2$
2. Case $\delta \neq 0$
3. Proof of the Liouville theorem case $\delta = 0$
4. Proof of the Liouville theorem case $\delta \neq 0$
Outline of the talk

1. Case $\delta = 0$, $(N - 2)p < N + 2$

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Existence of a Lyapunov functional:

$$\frac{d}{dt} E_0(u) = - \int_{\mathbb{R}^N} |\partial_t u|^2 dx$$

where

$$E_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx.$$ 

**Remark:** From Ball 77, we have $E(u_0) < 0 \Rightarrow u(t)$ blows up in finite time.
Extensive bibliography $\delta = 0$

- **Existence of Blow-up solutions?** yes, energy method by Levine 1974 and Ball 1977.

If $u$ blows up at time $T$, then

$$\forall t \in [0, T), \|u(t)\|_{L^\infty} \leq Cv(t),$$

with $v(t) = \kappa(T - t)^{-\frac{1}{p - 1}}, \kappa = (p - 1)^{-\frac{1}{p - 1}}$ and

$$\begin{cases} v'(t) = v(t)^p, \\ v(T) = +\infty. \end{cases}$$

**Definition:** We say that $u$ is of "type I".
Asymptotic Behavior (Blow-up profile $\delta = 0$)


Given a blow-up point $a$, the (supposed to be generic) profile is the following:

$$u(x, t) \sim (T - t)^{-\frac{1}{p-1}} f_0 \left( \left| \frac{x - a}{\sqrt{(T - t)|\log(T - t)|}} \right| \right),$$

where $f_0(z) = (p - 1 + b(p)z)^{-\frac{1}{p-1}}$. 

\[\hat{u}(x,t)\]
\[u(x,t)\]
\[\hat{a} + R[(T-t)|\log(T-t)|]^{1/2}\]
\[\kappa(T-t)^{-1/(p-1)}\]
\[x\]
\[\hat{a}\]
Remark: If $N = 1$, we know it is generic (Herrero, Velázquez). If $N \geq 2$, open problem.

- Stability of the blow-up profile ($\delta = 0$)

Theorem (Fermanian, Merle, Z. 2000) Consider initial data $\hat{u}_0$, the solution $\hat{u}(x, t)$ of (Equation) with blow-up time $\hat{T}$, blow-up point $\hat{a}$ and profile $f_0$ centered at $(\hat{T}, \hat{a})$.

Then, $\exists \mathcal{V}$ neighborhood of $\hat{u}_0$ s.t. $\forall u_0 \in \mathcal{V}$, $u(x, t)$ the solution of (Equation) blows up at time $T$, at a point $a$, with the profile $f_0$ centered at $(T, a)$.

Moreover, $(T, a) \rightarrow (\hat{T}, \hat{a})$ as $u_0 \rightarrow \hat{u}_0$.
A Liouville Theorem for equation (Equ\(u_0\))

**Theorem**

Assume that \(u\) is a solution of (Equ\(u_0\)) s.t.

\[
\forall (x, t) \in \mathbb{R}^N \times (-\infty, T), |u(x, t)| \leq M(T - t)^{-\frac{1}{p-1}}.
\]

Then,

\[u \equiv 0\] or \(\forall (x, t) \in \mathbb{R}^N \times (-\infty, T), u(x, t) = \pm \kappa (T_0 - t)^{-\frac{1}{p-1}},\]

for some \(T_0 \geq T\).
Consequences of the Liouville Theorem for equation \((\text{Eq} \, u_0)\)

**Proposition** Consider \(u\) a solution of \((\text{Eq} \, u_0)\), which blows up at time \(T\).
Then, (i) \((L^\infty\) estimates for \(u\) and derivatives)

\[
\|u(t)\|_{L^\infty(T-t)}^{\frac{1}{p-1}} \to \kappa \quad \text{and} \quad \|\nabla^k u(t)\|_{L^\infty(T-t)}^{\frac{1}{p-1} + \frac{k}{2}} \to 0
\]
as \(t \to T\) for \(k = 1, 2\) or 3.

(ii) (Uniform ODE localization) For all \(\varepsilon > 0\), there is \(C(\varepsilon)\) such that \(\forall x \in \mathbb{R}^N, \forall t \in [0, T),\)

\[
\left| \frac{\partial u}{\partial t}(x, t) - |u|^{p-1}u(x, t) \right| \leq \varepsilon |u(x, t)|^p + C.
\]

*Other consequences:* Regularity of the set of all blow-up points, see Z. 2006.
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1. Case $\delta = 0$, $(N - 2)p < N + 2$

2. Case $\delta \neq 0$

3. Proof of the Liouville theorem case $\delta = 0$

4. Proof of the Liouville theorem case $\delta \neq 0$


What is unknown? The blow-up rate, the blow-up profile, etc......

Our approach: Try to prove a Liouville Theorem.
A Liouville theorem for equation (Eq\( u_\delta \)), \( \delta \neq 0 \)

**Theorem (Nouaili,Z.)**

If \( 0 < |\delta| \leq \delta_0 \) and

\[
\forall (x, t) \in \mathbb{R}^N \times (-\infty, T) \ |u(x, t)| \leq M(\delta)(T - t)^{-\frac{1}{p-1}}
\]

for some \( \delta_0 > 0 \) and \( M(\delta) > 0 \), then,

\[
u \equiv 0 \text{ or } \forall (x, t) \in \mathbb{R}^N \times (-\infty, T), \ u(x, t) = \kappa e^{i\theta_0} (T_0 - t)^{-\frac{1+i\delta}{p-1}},
\]

for some \( T_0 \geq T \) and \( \theta_0 \in \mathbb{R} \).

**Rk.** \( M(\delta) \to +\infty \) as \( \delta \to 0 \).
**Uniform blow-up estimates**

**Proposition** Consider $0 < |\delta| \leq \delta_0$ and $u$ a solution of \((\text{Equ}_\delta)\) that blows up at time $T$ and satisfies

$$\forall t \in [0, T), \|u(t)\|_{L^\infty} \leq M(\delta)(T - t)^{-\frac{1}{p-1}}. \text{ (type I)}$$

Then, (i) \((L^\infty \text{ estimates for derivatives})\)

$$\|u(t)\|_{L^\infty}(T - t)^{-\frac{1}{p-1}} \to \kappa \text{ and } \|\nabla^k u(t)\|_{L^\infty}(T - t)^{\frac{1}{p-1} + \frac{k}{2}} \to 0$$

as $t \to T$ for $k = 1, 2$ or $3$.

(ii) \((\text{Uniform ODE localization})\) For all $\varepsilon > 0$, there is $C(\varepsilon)$ such that $\forall x \in \mathbb{R}^N$, $\forall t \in [0, T)$,

$$\left| \frac{\partial u}{\partial t}(x, t) - (1 + i\delta)|u|^{p-1}u(x, t) \right| \leq \varepsilon|u(x, t)|^p + C.$$

**Proof** It follows from the Liouville theorem.
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2. Case $\delta \neq 0$

3. Proof of the Liouville theorem case $\delta = 0$
   - Part 1: Limits of $w$ as $s \to \pm \infty$
   - Part 2: Trivial cases
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$
     - Step 1: Linearization of $w$ near $\kappa$ as $s \to -\infty$
     - Step 2: The relevant case, $\lambda = 1$
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

4. Proof of the Liouville theorem case $\delta \neq 0$
Let us recall the Liouville Theorem for:

\[ \partial_t u = \Delta u + |u|^{p-1} u. \]

**Theorem**

Assume that \( u \) is a solution of (Equ\(_0\)) s.t.

\[ \forall (x, t) \in \mathbb{R}^N \times (-\infty, T), \ |u(x, t)| \leq M(T - t)^{-\frac{1}{p-1}}. \]

Then,

\[ u \equiv 0 \text{ or } \forall (x, t) \in \mathbb{R}^N \times (-\infty, T), \ u(x, t) = \pm \kappa(T_0 - t)^{-\frac{1}{p-1}}, \]

for some \( T_0 \geq T \).
Case $\delta = 0, (N - 2)p < N + 2$
Proof of the Liouville theorem case $\delta = 0$

Proof of the Liouville theorem case $\delta \neq 0$

Part 1: Limits of $w$ as $s \to \pm \infty$

Part 2: Trivial cases

Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$

Statement in selfsimilar variables:

$$w_a(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t),$$

for all $(x, t) \in \mathbb{R}^N \times (-\infty, T)$, the function $w = w_a$ satisfies for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$:

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{(p - 1)} w + |w|^{p-1} w. \quad (Eqw_0)$$
**Theorem** (A Liouville theorem for equation (Eq $w_0$)) If

$$\|w(y, s)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})} \leq M$$

and $w$ is a solution of (Eq $w_0$), then

$$w \equiv 0 \text{ or } w \equiv \pm \kappa \text{ or } w = \pm \varphi_0(s - s_0),$$

for some $s_0 \in \mathbb{R}$, and

$$\varphi_0(s) = \kappa(1 + e^s)^{-\frac{1}{(p-1)}} \text{ and } \kappa = (p - 1)^{-\frac{1}{p-1}}.$$
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     - Step 1: Modulation
     - Step 2: Behavior as $s \to -\infty$
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     - Step 4: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
A Lyapunov functional in the $w$ variable

$$
\mathcal{E}(w) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho(y)dy
$$

with

$$
\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.
$$

Consequence: $w_{\pm\infty} = \lim_{s \to \pm\infty} w(y, s)$ exists and is a stationary solution of (Eq\,$w_0$). From Giga and Kohn we obtain $w_{\pm\infty} = 0$, $w_{\pm\infty} = \kappa$ or $w_{\pm\infty} = -\kappa$. 
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Since $\mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = \int_{-\infty}^{+\infty} ds \int_{\mathbb{R}} \left| \frac{\partial w}{\partial s}(y, s) \right|^2 dy \geq 0$

and $\mathcal{E}(\kappa) = \mathcal{E}(-\kappa) > 0 = \mathcal{E}(0)$,

we have 2 cases:

- **(Trivial)**
  \[ \mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) = 0 \Rightarrow \partial_s w \equiv 0 \Rightarrow w \equiv 0 \text{ or } w \equiv \pm \kappa. \]

- **(Non trivial)**
  \[ \mathcal{E}(w_{-\infty}) - \mathcal{E}(w_{+\infty}) > 0 \Rightarrow (w_{-\infty}, w_{+\infty}) \equiv (\pm \kappa, 0). \]
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Step 1: Linearization of \( w \) near \( \kappa \) as \( s \to -\infty \)

We consider \( v(y, s) = w(y, s) - \kappa \).

\[
\partial_s v = \mathcal{L}v + f(v), \text{ with } \mathcal{L}v = \Delta v - \frac{1}{2}y \cdot \nabla v + v, \quad |f(v)| \leq C|v|^2.
\]

\( \mathcal{L} \) is self-adjoint, \( \text{spec}(\mathcal{L}) = \{1 - \frac{m^2}{2} | m \in \mathbb{R} \} \).

The eigenvectors are Hermite polynomials.

As \( s \to -\infty \), one of the following cases occurs:

- i) \( \lambda = 1 \), \( w(y, s) = \kappa + C_0 e^s + o(e^s), \) \( C_0 \in \mathbb{R} \).
- ii) \( \lambda = \frac{1}{2} \), \( w(y, s) = \kappa + C_1 e^{s/2} y + o(e^{s/2}), \) \( C_1 \in \mathbb{R}^* \).
- iii) \( \lambda = 0 \), \( w(y, s) = \kappa - \frac{\kappa}{2ps}(\frac{1}{2}y^2 - 1) + o(\frac{1}{s}). \)

Convergence is in \( L_\rho^2 \) and uniformly on compact sets.
Step 2: The relevant case, $\lambda = 1$

If $\varphi^*(s) = \begin{cases} 
\kappa & \text{if } C_0 = 0, \\
\varphi_0(s - s_0) = \kappa (1 + e^{s-s_0})^{-\frac{1}{p-1}} & \text{if } C_0 < 0, \\
\tilde{\varphi}(s - s_0) = \kappa (1 - e^{s-s_0})^{-\frac{1}{p-1}} & \text{if } C_0 > 0,
\end{cases}$

with $s_0 = -\log \left( \frac{(p-1)}{\kappa} |C_0| \right)$, then $\varphi^*$ is a solution of (Eq$w_0$) with the same expansion of $w$ as $s \to -\infty$.

If $V = w - \varphi^*$, then $\| V(y, s) \|_{L^2_\rho} = O(e^{3/2s})$.

Since $\frac{3}{2} > 1 = \max \{ \lambda \in \text{spec}(L) \}$, then $V \equiv 0$.
Because $w_{-\infty} = 0$, we get $\varphi^* = \varphi_0(s - s_0)$.

$$w(y, s) = \varphi(s - s_0) = \kappa (1 + e^{s-s_0})^{-\frac{1}{p-1}},$$ for some $s_0 \in \mathbb{R}$.
Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

Merle-Zaag (Blow-up criterion). Let $W$ a solution of $(Eqw_0)$, such that

$$
\left( \int |W(y, s_0)|^2 \rho(y) dy \right)^{\frac{p+1}{2}} > 2^{\frac{p+1}{p-1}} \mathcal{E}(W(., s_0)), \quad (Is_0)
$$

for some $s_0 \in \mathbb{R}$. Then $W$ blows-up at some time $S > s_0$.

In case ii) and iii) one can find $a_0$ and $s_0$ such that $(Is_0)$ is true with $W(y, s_0) = w_{a_0}(y, s_0) = w(y + a_0 e^{s_0/2}, s_0)$.

Then, there exists $S > s_0$, such that $w_{a_0}$ blows up at $S$, contradiction because $w (w(y, s) = w_{a_0}(y - a_0 e^{s/2}, s))$ is bounded.
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A Liouville theorem for vector valued semilinear heat equations with...
No Lyapunov functional:

- No Lyapunov functional to get the limits as $s \to \pm \infty$.
- No blow-up criterion to rule out the irrelevant cases.
Let us recall the Liouville Theorem for:

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1}u.$$ 

**Theorem (Nouaili, Z.)**

If $0 < |\delta| \leq \delta_0$ and $u$ is a solution of $(\text{Equ}_\delta)$ satisfying

$$\forall (x, t) \in \mathbb{R}^N \times (-\infty, T) \ |u(x, t)| \leq M(\delta)(T - t)^{-\frac{1}{p-1}}$$

for some $\delta_0 > 0$ and $M(\delta) > 0$, then,

$$u \equiv 0 \text{ or } \forall (x, t) \in \mathbb{R}^N \times (-\infty, T), \ u(x, t) = \kappa e^{i\theta_0}(T_0 - t)^{-\frac{1+i\delta}{p-1}},$$

for some $T_0 \geq T$ and $\theta_0 \in \mathbb{R}$. 

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Hatem ZAAG  
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A Liouville theorem for vector valued semilinear heat equations with
Statement in selfsimilar variables:

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\begin{align*}
  w_a(y, s) &= (T - t)^{1 + i\delta \over p - 1} u(x, t), \\
  y &= \frac{x - a}{\sqrt{T - t}}, \\
  s &= -\log(T - t),
\end{align*}
\]

for all \((x, t) \in \mathbb{R}^N \times (-\infty, T)\), the function \(w = w_a\) satisfies for all \((y, s) \in \mathbb{R}^N \times \mathbb{R}\):

\[
  w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1 + i\delta}{(p - 1)} w + (1 + i\delta)|w|^{p - 1} w. \quad (\text{Eq} w_\delta)
\]
**Theorem** (A Liouville theorem for equation (Eq$w_{\delta}$)) If $0 < |\delta| \leq \delta_0$ and $w$ is a solution of (Eq$w_{\delta}$) s.t.

$$
\|w(y, s)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})} \leq M(\delta),
$$

then,

$$w \equiv 0 \text{ or } w \equiv \kappa e^{i\theta_0} \text{ or } w = \varphi_{\delta}(s - s_0)e^{i\theta_0},$$

for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where

$$
\varphi_{\delta}(s) = \kappa(1 + e^s)^{(1+i\delta)/(p-1)} \text{ and } \kappa = (p - 1)^{-1/(p-1)}.
$$
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**Case $\delta = 0$, $(N - 2)p < N + 2$**

**Case $\delta \neq 0$**

Proof of the Liouville theorem case $\delta = 0$

Proof of the Liouville theorem case $\delta \neq 0$

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*(Stationary solution)* Consider $w \in L^\infty(\mathbb{R}^N)$ a stationary solution of (Eq\$w_\delta\$). Then, $w \equiv 0$ or there exists $\theta_0 \in \mathbb{R}$ such that $w \equiv \kappa e^{i\theta_0}$.

**Remark:** The proof is trivial and much easier than the case $\delta = 0$.

To get the limits, we have no Lyapunov functional.

Fortunately, a perturbation method used by Andreucci, Herrero and Velázquez, works here and yields the following:

**Proposition** If $0 < |\delta| \leq \delta_0$ and $w$ is a solution of (Eq\$w_\delta\$) satisfying for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $|w(y, s)| \leq M(\delta)$ for some $\delta_0$ and $M(\delta)$, then, as $s \to -\infty$

- either 
  $(i) \quad \|w(., s)\|_{L^2_\rho} \to 0$

- or 
  $(ii) \quad \inf_{\theta \in \mathbb{R}} \|w(., s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$. 

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   - Part 1: Limits of $w$ as $s \to -\infty$
   - Part 2: Case where $w \to 0$ as $s \to -\infty$
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \| w(\cdot, s) - \kappa e^{i\theta} \|_{L^2} \to 0$ as $s \to -\infty$
If $h(s) \equiv \int_{\mathbb{R}} |w(y, s)|^2 \rho(y) dy$, then

$$h'(s) \leq -\frac{2}{p-1} h(s) + 2 \int_{\mathbb{R}} |w(y, s)|^{p+1} \rho(y) dy.$$  

Using the regularizing effect of equation (Eq $w_{\delta}$), we derive the following delay estimate, for some positive $s^*$ and $C$

$$\forall s \in \mathbb{R}, \quad h'(s) \leq -\frac{2}{p-1} h(s) + C(M) h(s-s^*) \frac{p+1}{2}.$$  

Using $h(s) \to 0$ as $s \to -\infty$ and delay ODE techniques, we have for some $\varepsilon > 0$ small enough,

$$\forall \sigma \in \mathbb{R}, \forall s \geq \sigma + s^*, \quad h(s) \leq \varepsilon e^{-\frac{2(s-\sigma)}{p-1}},$$

Fixing $s$ and letting $\sigma \to -\infty$, we get $w \equiv 0$.  

Hatem ZAAG, LAGA, CNRS UMR 7539 Université Paris 13  

A Liouville theorem for vector valued semilinear heat equations with
Outline of the talk

1. Case $\delta = 0$, $(N - 2)p < N + 2$  
   - Part 1: Limits of $w$ as $s \to \pm \infty$  
   - Part 2: Trivial cases  
   - Part 3: Case when $w_{-\infty} \to \kappa$ as $s \to -\infty$  
     - Step 1: Linearization of $w$ near $\kappa$ as $s \to -\infty$  
     - Step 2: The relevant case, $\lambda = 1$  
     - Step 3: The irrelevant cases; ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$

2. Case $\delta \neq 0$  

3. Proof of the Liouville theorem case $\delta = 0$  
   - Part 1: Limits of $w$ as $s \to \pm \infty$  
   - Part 2: Case where $w \to 0$ as $s \to -\infty$  
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \| w(\cdot, s) - \kappa e^{i\theta} \|_{L^p} \to 0$ as $s \to -\infty$

4. Proof of the Liouville theorem case $\delta \neq 0$  
   - Part 1: Limits of $w$ as $s \to -\infty$  
   - Part 2: Case where $w \to 0$ as $s \to -\infty$  
   - Part 3: Case where $\inf_{\theta \in \mathbb{R}} \| w(\cdot, s) - \kappa e^{i\theta} \|_{L^p} \to 0$ as $s \to -\infty$
     - Step 1: Modulation  
     - Step 2: Behavior as $s \to -\infty$  
     - Step 3: The relevant case $\lambda = 1$  
     - Step 4: The irrelevant cases, ii) $\lambda = \frac{1}{2}$ or iii) $\lambda = 0$
Step 1: Modulation

We introduce $\theta(s)$ and $v$ such that

$$w(y, s) = e^{i\theta(s)}(v(y, s) + \kappa), \quad \forall s \leq s_1, \quad \int (\text{Im } v - \delta \text{Re } v) \rho = 0. (*)$$

$$\partial_s v = \tilde{L} v - i \theta_s (v + \kappa) + G,$$

where

$$\tilde{L} v = \Delta v - \frac{1}{2} y \nabla v + (1 + i \delta) v_1, \quad |G(v)| \leq C|v|^2.$$

$\text{spec}(\tilde{L}) = \{1 - \frac{m}{2} |m \in \mathbb{R}\}$ its eigenvectors are given by

$$\{ (1 + i \delta) h_m, i h_m | n \in \mathbb{N} \}$$

and $h_m$ are Hermite polynomials.

The choice of $\theta(s) \ (\ast)$ kills one neutral mode.
Step 2: Behavior as $s \to -\infty$

- $\lambda = 1$, with eigenfunction $(1 + i\delta)h_0(y) = (1 + i\delta)$.
- $\lambda = 1/2$, with eigenfunction $(1 + i\delta)h_1(y) = (1 + i\delta)y$.
- $\lambda = 0$, with two eigenfunctions $(1 + i\delta)h_2(y) = (1 + i\delta)(y^2 - 2)$ and $ih_0(y) = i$ (killed by the choice of $\theta(s)$ ($\star$)).

We have one of the following cases as $s \to -\infty$:

(i) $w(y, s) = \{\kappa + (1 + i\delta)C_0e^s\}e^{i\theta_0} + o(e^{3s}), \ C_0 \in \mathbb{R}$
(ii) $w(y, s) = \{\kappa + (1 + i\delta)C_1e^{s/2}y\}e^{i\theta_0} + o(e^{s/2}), \ C_1 \in \mathbb{R}^*$,
(iii) $w(y, s) = e^{i\theta_0}\{\kappa - (1 + i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2 - 2) - i\frac{(1+\delta^2)\delta^2\kappa^2}{2(p-\delta^2)^2} \frac{1}{s}\} + o\left(\frac{1}{|s|}\right)$.

Convergence takes place in $L^2_\rho$ and uniformly on compact sets.
Step 3: The relevant case, $\lambda = 1$

We do exactly as in case $\delta = 0$.

If $\varphi^*(s) = \begin{cases}  
= \kappa e^{i\theta_0} & \text{if } C_0 = 0, \\
= \varphi_{\delta}(s - s_0) = \kappa e^{i\theta_0} (1 + e^{s-s_0})^{-\frac{1+i\delta}{p-1}} & \text{if } C_0 < 0, \\
= \tilde{\varphi}_{\delta}(s - s_0) = \kappa e^{i\theta_0} (1 - e^{s-s_0})^{-\frac{1+i\delta}{p-1}} & \text{if } C_0 > 0,
\end{cases}$

with $s_0 = -\log \left( \frac{(p-1)}{\kappa} |C_0| \right)$ and $\theta_0 \in \mathbb{R}$. Then $\varphi^*$ is a solution of $(Eq\varphi_{\delta})$ with the same expansion of $w$ as $s \to -\infty$.

If $V = w - \varphi^*$, then $\|V(y, s)\|_{L^2_p} = O(e^{3/2s})$.

Since $\frac{3}{2} > 1 = \max\{\lambda \in \text{spec}(\mathcal{L})\}$, then $V \equiv 0$.

Because $w$ is bounded, we get $\varphi^* \neq \tilde{\varphi}_{\delta}$, hence $w(y, s) = \kappa e^{i\theta_0}$ or $w(y, s) = \varphi_{\delta}(s - s_0)e^{i\theta_0} = \kappa e^{i\theta_0} (1 + e^{s-s_0})^{-\frac{1+i\delta}{p-1}}$, for some $s_0 \in \mathbb{R}$.
Step 4: The irrelevant cases, ii) \( \lambda = \frac{1}{2} \) or iii) \( \lambda = 0 \)

No blow-up criterion? Our source of inspiration is Velázquez’s work.
We extend the convergence in ii) and iii) from \(|y| < R\) to larger regions to find singular profiles.

\[
\text{ii) } f_1(\xi) = \kappa(1 - C_1\kappa^{-p}\xi)^{-\frac{(1+i\delta)}{p-1}} \text{ singular for } \xi = R_1(p) \\
\lim_{s \to -\infty} \sup_{|y| \leq Re^{-s/2}} \left| w(y, s) - f_1(ye^{s/2}) \right| = 0, \text{ where } R < R_1(p).
\]

\[
\text{iii) } f_2(\xi) = \kappa \left(1 - \frac{(p - 1)}{4(p - \delta^2)}\xi^2 \right)^{-\frac{(1+i\delta)}{p-1}} \text{ singular for } \xi = R_2(p) \\
\lim_{s \to -\infty} \sup_{|y| \leq R\sqrt{-s}} \left| w(y, s) - f \left(\frac{y}{\sqrt{-s}}\right) \right| = 0 \text{ where } R < R_2(p).
\]
Here, we choose \( R = R(M) \) such that \( |f_2\left(\frac{R}{\sqrt{-s}}\right)| = 2M \), where

\[ \|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq M = M(\delta) \quad (*) \].

Then, for \(|s|\) large enough,

\[ |w(R\sqrt{-s}, s) - f_2\left(\frac{R}{\sqrt{-s}}\right)| \leq \frac{M}{2}, \quad |w(R\sqrt{-s}, s)| \geq |f_2\left(\frac{R}{\sqrt{-s}}\right)| - \frac{M}{2} = \frac{3M}{2} \].

Contradiction with (*)

Hatem ZAAG, LAGA, CNRS UMR 7539 Université Paris 13