Similarities in blow-up approaches between semilinear heat and wave equations

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This short note is intended to non specialists. We aim at showing a surprising fact: how a “parabolic” program initially developed for blow-up solutions of the *semilinear* heat equation works for blow-up solutions of the *semilinear* wave equation. We feel this fact surprising because for the *linear* equations, everything separates the heat and the wave equations.

For simplicity, all equations are considered on the whole space $\mathbb{R}^N$. 

Differences between linear equations

*Heat equation* $\partial_t u = \Delta u$
- Regularizing effect
- Infinite speed of propagation
- Dissipation of the energy $\int |\nabla u|^2dx$, non reversible equation

*Wave equation* $\partial^2_{tt} u = \Delta u$
- No gain of regularity
- Finite speed of propagation ($c = 1$)
- Conservation of the energy $\int \left( |\partial_t u|^2 + |\nabla u|^2 \right) dx$, reversible equation
Semilinear equations

the heat: $\partial_t u = \Delta u + |u|^{p-1}u$ where $p > 1$ is subcritical with respect to the Sobolev injection:

$$p < 1 + \frac{4}{N-2} \quad \text{if } N \geq 3.$$ 

the wave: $\partial_{tt}^2 u = \Delta u + |u|^{p-1}u$ where $p > 1$ is subcritical with respect to the conformal invariance:

$$p \leq 1 + \frac{4}{N-1} \quad \text{if } N \geq 2.$$
Solution of the Cauchy problem and existence of blow-up solutions

The maximal solution either exists for all \( t > 0 \) (global solution) or on \([0, T)\) for some \( T > 0 \). In that case:

the heat: \( \|u(t)\|_{L^\infty} \to \infty \) as \( t \to T \),

the wave: \( \|u(t)\|_{L^2_{\text{loc}, u}} + \|u(t)\|_{L^2_{\text{loc}, u}} + \|u(t)\|_{L^2_{\text{loc}, u}} \to \infty \) as \( t \to T \) where \( L^2_{\text{loc}, u} \) is the set of all \( v \) such that

\[
\|v\|_{L^2_{\text{loc}, u}}^2 \equiv \sup_{a \in \mathbb{R}^N} \int_{|x-a|<1} |v(x)|^2 \, dx < +\infty.
\]
**Remark:** For the semilinear heat equation, no matter how weak is the initial regularity (let’s stay in $L^q$ spaces), blow-up occurs always in $L^\infty$ due to the regularizing effect. See Weissler [10]. For the wave equation, there is no regularizing effect. We work with weak solutions and consider the case where $u$, $\partial_t u$ and $\nabla u$ are in $L^{2}_{\text{loc},u}$. 
Trivial solutions

When initial data do not depend on space, we just have to solve an ODE. We have the following solutions

the heat: $v' = v^p$ whose solution is $v(t) = \kappa_h(T - t)^{-\frac{1}{p-1}}$ where $\kappa_h = (p - 1)^{-\frac{1}{p-1}}$ for any $T > 0$.

the wave: $v'' = v^p$ whose solution is $v(t) = \kappa_w(T - t)^{-\frac{2}{p-1}}$ where $\kappa_w = \left(\frac{2(p+1)}{(p-1)^2}\right)^{-\frac{1}{p-1}}$ for any $T > 0$. 
**Question:** Take an arbitrary solution which blows up at time $T$. Can we estimate its blow-up rate? More precisely, can we find an equivalent of its norm in the Cauchy space?

**Answer** (the same for the heat and the wave): the blow-up rate is given by the solution of the associated ODE which blows-up at the same time $T$. More precisely,
Heat equation

**Theorem (A result due to Giga and Kohn [2] and [3] and Giga, Matsui and Sasayama [4]).** Let $u$ be a solution of $\partial_t u = \Delta u + |u|^{p-1}u$ where $p > 1$ and $p < 1 + 4/(N - 2)$ if $N \geq 3$ which blows up at time $T > 0$. Then, for all $t \in [0, T)$,

$$\kappa_h(T - t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(T - t)^{-\frac{1}{p-1}}$$

where $C = C(\|u_0\|, T)$.

Near the blow-up time, we have this better estimate:

**Theorem (Merle and Zaag [6] and [5], see also the note [7]).** Under the same hypotheses,

$$\|u(t)\|_{L^\infty} \sim \kappa_h(T - t)^{-\frac{1}{p-1}} \text{ as } t \to T.$$
Wave equation

Theorem (Merle and Zaag [8] and [9]). Let $u$ be a solution of
$$
\partial_{tt}^2 u = \Delta u + |u|^{p-1}u \quad \text{where } p > 1 \text{ and } p \leq 1 + 4/(N-1) \text{ if } N \geq 2
$$
which blows up at time $T > 0$. Then, for all $t > 0$,

$$
\epsilon_{N,p} \leq (T-t)^{\frac{2}{p-1}} \|u(t)\|_{L^2_{\text{loc},u}}
+ (T-t)^{\frac{2}{p-1}+1} \left( \|\nabla u(t)\|_{L^2_{\text{loc},u}} + \|\partial_t u(t)\|_{L^2_{\text{loc},u}} \right) \leq C.
$$

Remark: In both cases (heat and wave), lower bounds on the blow-up rate are trivial.
A common method for the proof: the self-similar change of variables

Let $u$ be a solution that blows up at time $T > 0$. For each $a \in \mathbb{R}^N$, we introduce $w_a(y, s)$ defined by:

heat equation:

$$w_a(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x - a}{\sqrt{T - t}}, \quad s = - \log(T - t).$$

wave equation:

$$w_a(y, s) = (T - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - a}{T - t}, \quad s = - \log(T - t).$$
Remark: In both cases, \( w_a \) is the ratio between the blow-up solution \( u \) and its supposed blow-up rate (given by the ode). Hence, the goal is to show that for all \( s \geq - \log T \),

\[
\frac{1}{C_0} \leq \| w(s) \| \leq C_0.
\]

Remark: In both cases, studying the behavior of \( u(x, t) \) when \( (x, t) \) approaches \( (a, T) \) is equivalent to the study of the long-time asymptotics of \( w_a(y, s) \) when \( y \) is near 0 and the new time variable \( s \) goes to infinity.

Remark: In both cases, the new space variable \( y \) is a time dependent zoom of the old one \( x \) near the point \( a \). This zoom becomes sharper as \( t \to T \) (that is \( s \to \infty \)). However, \( y \) is not the same in the heat and the wave setting, since in the former, a derivative in space is like half a derivative in time, whereas in the latter, space and time play the same role.
Equations satisfied by $w_a(y, s)$ (or $w(y, s)$ for simplicity)

For all $y \in \mathbb{R}^N$ and $s \geq -\log T$,

the heat:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1}w,$$

the wave:

$$\partial_{ss}^2 w - \text{div} (\nabla w - (y \cdot \nabla w)y) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1}w$$

$$= -\frac{p+3}{p-1} \partial_s w - 2y \cdot \nabla \partial_s w.$$

Remark: Surprisingly, the new wave equation is dissipative, unlike the original. This means that we are unveiling a new structure in the problem.
A new structure derived from the self-similar transformation: a Lyapunov functional: the heat (Giga and Kohn [2])

If

\[ E_h(w) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla w|^2 + \frac{1}{2(p - 1)} - \frac{1}{p + 1} |w|^{p+1} \right) \exp(-|y|^2/4) \, dy, \]

then

\[ \frac{d}{ds} E_h(w(s)) = - \int_{\mathbb{R}^N} (\partial_s w)^2 \exp(-|y|^2/4) \, dy. \]
A new structure derived from the self-similar transformation: a Lyapunov functional: the wave (Antonini and Merle [1])

If

\[ E_w(w) = \int_{B(0,1)} \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2}(y.\nabla w)^2 
+ \frac{(p + 1)}{(p - 1)^2} w^2 - \frac{1}{p + 1}|w|^{p+1} \right) (1 - |y|^2)^\alpha dy \]

where \( \alpha = \frac{2}{p-1} - \frac{N-1}{2} \geq 0 \), then

\[ \frac{d}{ds} E_w(w(s)) = -2\alpha \int_{B(0,1)} (\partial_s w)^2 (1 - |y|^2)^{\alpha-1} \text{ when } p < 1 + \frac{4}{N - 1}. \]
Remark: The natural space domain in the wave setting is the unit ball which corresponds in the \((x, t)\) variable to the backward light cone with vertex \((a, T)\), a notion adapted to the finite speed of propagation \((c = 1)\).

Remark: For the wave equation, when \(p = 1 + \frac{4}{N-1}\) (critical case), the dissipation of \(E_w\) becomes degenerate and is supported in the boundary of the unit ball.

Remark: Still for the wave equation, please note that this Lyapunov functional is not the energy in conformal coordinates.
The same blow-up criterion in similarity variables

Prop. If a solution $W$ satisfies $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then $W$ blows up in finite time $S^* > s_0$.

Proof: For the heat, this is classical. For the wave, see Antonini and Merle [1]).

Since all $w_a(y, s)$ are defined for all $s \geq -\log T$ by construction, they never blow-up. More precisely, we have the following:

Corollary. For all $a \in \mathbb{R}^N$ and $s \geq -\log T$,

$$E(w_a(s)) \geq 0.$$ 

Remark: $E$ stands for $E_h$ or $E_w$ here.
Control of the energy

Because of the blow-up criterion and the monotonicity of $E$, it holds that
\[ \forall a \in \mathbb{R}^N, \quad \forall s \geq - \log T, \quad 0 \leq E(w_a(s)) \leq E(w_a(- \log T)) \leq C_0(T, \|u_0\|). \]

End of the proof:

Since the energy is bounded, one has to use interpolation in Sobolev spaces and show that each term in the energy is bounded, uniformly with respect to the scaling point $a$. See Giga and Kohn [3] and Giga, Matsui and Sasayama [4] for the heat; see Merle and Zaag [8] and [9] for the wave equation.
References


