Determination of the blow-up rate for the semilinear wave equation

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Abstract : In this paper, we find the optimal blow-up rate for the semilinear wave equation with a power nonlinearity. The exponent p is superlinear and less than $1+\frac{4}{N-1}$ if $N \geq 2$.

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1 introduction

We are concerned in this paper with blow-up solutions for the following semilinear wave equation

$$\begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$. The space $\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ is the set of all v in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$ such that

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})} \equiv \sup_{a\in\mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx\right)^{1/2} < +\infty.$$

The space $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ is the set of all v in $\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ such that $\nabla v \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$. We assume in addition that

$$1 < p < 1 + \frac{4}{N-1}. \tag{2}$$

The Cauchy problem for equation in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2(\mathbb{R}^N)$. See for instance Lindblad and Sogge [11], Shatah and Struwe [13] and their references (for the local in time wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2(\mathbb{R}^N)$). The existence of blow-up solutions for equation (1) is a consequence of the finite speed of propagation and ODE techniques (see for example John [8]). More blow-up results can be found in Caffarelli and Friedman [3], Alinhac [1], Kichenassamy and Litman [9], [10]. Given a solution u of (1) that blows up at time $T > 0$, we aim at controlling its blow-up norm in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$. More precisely, we would like to compare the growth of u with the growth of v, a solution of the associated ODE :

$$v_{tt} = v^p, \quad v(T) = +\infty,$$

that is $v(t) \sim \kappa (T-t)^{-\frac{2}{p-1}}$ where $\kappa = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$. For this purpose, we introduce for each $a \in \mathbb{R}^N$ the following self-similar change of variables :

$$w_a(y,s) = (T-t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x-a}{T-t}, \quad s = -\log(T-t). \tag{3}$$

The function w_a (we write w for simplicity) satisfies the following equation for all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$w_{ss} + \frac{p+3}{p-1} w_s + 2y.\nabla w_s + \sum_{i,j} (y_i y_j - \delta_{i,j}) \partial^2_{y_i y_j} w + \frac{2(p+1)}{p-1} y.\nabla w = |w|^{p-1} w - \frac{2(p+1)}{(p-1)^2} w,$$

or in divergence form:

$$w_{ss} - \frac{1}{\rho} \operatorname{div}\left(\rho \nabla w - \rho (y.\nabla w) y\right) + \frac{2(p+1)}{(p-1)^2} w - |w|^{p-1} w = -\frac{p+3}{p-1} w_s - 2y.\nabla w_s \tag{4}$$

$$\text{where } \rho(y) = (1-|y|^2)^{\alpha} \text{ and } \alpha = \frac{2}{p-1} - \frac{N-1}{2} > 0. \tag{5}$$

Note that $\alpha > 0$ is equivalent to the condition $p < 1 + \frac{4}{N-1}$ stated in (2). Note also that s goes to infinity as t goes to T.

Caffarelli and Friedman have obtained in [3] results on blow-up solutions for equation (1), when a monotony condition is satisfied by the solution and $N = 1$. Antonini and Merle [2] have proved under some restrictions on the power p that all *positive* solutions of (4) are bounded in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$, which yields a growth estimate for *positive* blow-up solutions of (1). Their method strongly depends on positivity, since it relies on the nonexistence of positive solutions for

$$\Delta u + u^p = 0$$

in $\mathbb{R}^N$, if $p > 1$ and $(N-2)p < N+2$, as proved by Gidas and Spruck [4].

In this paper, we remove the positivity condition and prove the same result for unsigned solutions.

Theorem 1 (Uniform bounds on solutions of (4)) *If u is a solution of (1) that blows up at time T, then for all $s \geq -\log T + 1$,*

$$\epsilon_0 \leq \sup_{a\in\mathbb{R}^N} \|w_a(s)\|_{\mathrm{H}^1(B)} + \|\partial_s w_a(s)\|_{\mathrm{L}^2(B)} \leq K$$

where w_a is defined in (3), B is the unit ball of $\mathbb{R}^N$, $\epsilon_0 = \epsilon_0(N,p) > 0$, and K depends only on N, p and on bounds on T and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

Remark: The critical value for p in our theorem ($p = 1 + \frac{4}{N-1}$) is also critical for the existence of a conformal transformation for equation (1). Note that the Lyapunov functional E is the $w(y,s)$ variable is not the energy of the conformal transformation of u.
Remark: Let us remark that the lower bound in the theorem follows by standard techniques from scaling arguments and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2(\mathbb{R}^N)$. Indeed, let us assume by contradiction that there exists $s^* \geq -\log T + 1$ such that

$$\text{for all } a \in \mathbb{R}^N, \quad \|w_a(s^*)\|_{\mathrm{H}^1(B)} + \|\partial_s w_a(s^*)\|_{\mathrm{L}^2(B)} \leq \epsilon_0$$

where ϵ_0 will be fixed small. Let $t^* = T - e^{-s^*}$. We define for all $a \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ and $\tau \in [-\frac{t^*}{T-t^*}, 1)$,

$$v_a(\xi,\tau) = (T-t^*)^{\frac{2}{p-1}} u(a + \xi(T-t^*), t^* + \tau(T-t^*)).$$

The function v_a is a solution of equation (1) that blows up at time $\tau = 1$. Moreover,

$$\|v_a(0)\|_{\mathrm{H}^1(B(0,2))} + \|\partial_\tau v_a(0)\|_{\mathrm{L}^2(B(0,2))} \leq C\epsilon_0.$$

Using the finite speed of propagation and the local in time wellposedness in H^1 for equation (1), we obtain for some $M > 0$

$$\forall a \in \mathbb{R}^N, \quad \limsup_{\tau\to 1} \|v_a(\tau)\|_{\mathrm{H}^1(B(0,2))} + \|\partial_\tau v_a(\tau)\|_{\mathrm{L}^2(B(0,2))} \leq M,$$

which implies that

$$\lim_{t\to T} \|(u, \partial_t u)\|_{\mathrm{H}^1_{\mathrm{loc}} \times \mathrm{L}^2_{\mathrm{loc}}} \leq M'.$$

This contradicts the fact that T is a blow-up time for u. Therefore, the main point of the theorem is the existence of the constant K.
Note that our result remains true with the unit ball B replaced by $B(R)$, for any $R > 0$ (in that case, K depends also on R).
Remark: The result holds in the vector valued case with the same proof. Note that our proof strongly relies on the fact that α is positive. In particular, we don't give any answer in the range of subcritical exponent $1 + \dfrac{4}{N-1} \leq p < 1 + \dfrac{4}{N-2}$.
Remark: Note that a similar structure exists in the diffusive case (nonlinear heat equations) as has been exhibited and used by Giga and Kohn [5] to obtain uniform bounds in the similarity variables. Further refinements has been accomplished by Quittner [12] and Giga, Matsui and Sasayama [6].

As in [2], this theorem can be restated in the original set of variables $u(x,t)$:

Theorem 1' (Uniform bounds on blow-up solutions of equation (1)) *If u is a solution of (1) that blows up at time T, then for all $t \in [T(1-e^{-1}), T)$,*

$$\epsilon_0 \le (T-t)^{\frac{2}{p-1}} \|u\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)} + (T-t)^{\frac{2}{p-1}+1} \left(\|u_t\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)} + \|\nabla u\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)} \right) \le K$$

for some $\epsilon_0 \equiv \epsilon_0(N,p) > 0$ and a constant K which depends only on N, p and on bounds on T and the initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

The proof of the main result relies on:

- the existence of a Lyapunov functional for equation (4) and some energy estimates related to this structure,
- the improvement of regularity estimates by interpolation,
- some Gagliardo-Nirenberg type argument similar to that used once for nonlinear Schrödinger equation, where uniform H^1 bounds have been derived from L^2 and energy conservation in the subcritical case $p < 1 + \dfrac{4}{N}$ (see Ginibre and Velo [7]).

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2 Local energy estimates

2.1 A Lyapunov functional for equation (4)

We recall in this subsection some results from Antonini and Merle [2]. Throughout this section, w stands for any w_a defined in (3). As a matter of fact, all estimates we get are independent of $a \in \mathbb{R}^N$.

Antonini and Merle [2] showed that equation (4) had a Lyapunov functional defined by

$$E(w) = \int_B \left(\frac{1}{2} w_s^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} (y.\nabla w)^2 + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy \qquad (6)$$

where B is the unit ball of $\mathbb{R}^N$. More precisely, they have proved the following identity:

Lemma 2.1 *For all s_1 and s_2,*

$$E(w(s_2)) - E(w(s_1)) = -2\alpha \int_{s_1}^{s_2} \int_B w_s(y,s)^2 (1-|y|^2)^{\alpha-1} dy ds.$$

The authors have showed the following blow-up criterion for equation (4):

Lemma 2.2 (Blow-up criterion for equation (4)) *If a solution W of equation (4) satisfies $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S^* > s_0$.*

Since w is by definition defined for all $s \ge -\log T$, we get the following bounds:

Corollary 2.3 (Bounds on E) *For all $s \geq -\log T$, $s_2 \geq s_1 \geq -\log T$, the following identities hold:*

$$0 \leq E(w(s)) \leq E(w(-\log T)) \leq C_0, \tag{7}$$

$$\int_{s_1}^{s_2}\int_B w_s(y,s)^2(1-|y|^2)^{\alpha-1}dyds \leq \frac{C_0}{2\alpha}, \tag{8}$$

where C_0 depends only on bounds on T and the initial data of (1) in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

From now on, we adopt a strategy different from that of [2].

2.2 Space-time estimates for w

The space-time estimates we obtain in this section involve two relations between three different quantities

$$\int_{s_1}^{s_2}\int_B w^2\rho dyds, \quad \int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds \text{ and } \int_{s_1}^{s_2}\int_B |\nabla w|^2(1-|y|^2)\rho dyds,$$

where $1 \leq s_2 - s_1 \leq 3$. Let us first derive the two relations.

The first is obtained by integrating in time between s_1 and s_2, the expression (6) of $E(w)$:

$$\begin{aligned}\int_{s_1}^{s_2} E(w(s))ds &= \int_{s_1}^{s_2}\int_B \left(\frac{1}{2}w_s^2 + \frac{(p+1)}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dsdy\\ &+ \frac{1}{2}\int_{s_1}^{s_2}\int_B \left(|\nabla w|^2 - (y.\nabla w)^2\right)\rho dsdy.\end{aligned} \tag{9}$$

We derive the second relation by multiplying the equation (4) by $w\rho$ and integrating both in time and space over $B \times (s_1, s_2)$. After some straightforward integration by parts that we leave to Appendix A, we obtain the following identity:

$$\begin{aligned}&\left[\int_B \left(ww_s + \left(\frac{p+3}{2(p-1)} - N\right)w^2\right)\rho dy\right]_{s_1}^{s_2}\\ &+ \int_{s_1}^{s_2}\int_B \left(-w_s^2 - 2w_s y.\nabla w + |\nabla w|^2 - (y.\nabla w)^2\right)\rho dyds\\ &- 2\int_{s_1}^{s_2}\int_B w_s w y.\nabla\rho dyds = \int_{s_1}^{s_2}\int_B \left(|w|^{p+1} - \frac{2(p+1)}{(p-1)^2}w^2\right)\rho dyds.\end{aligned} \tag{10}$$

Using (10) to eliminate the second line in the energy integral (9), we obtain

$$\begin{aligned}\frac{(p-1)}{2(p+1)}\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds &= \int_{s_1}^{s_2} E(w(s))ds\\ &+ \int_{s_1}^{s_2}\int_B \left(-w_s^2\rho - w_s y.\nabla w\rho - w_s w y.\nabla\rho\right)dyds\\ &+ \frac{1}{2}\left[\int_B \left(ww_s + \left(\frac{p+3}{2(p-1)} - N\right)w^2\right)\rho dy\right]_{s_1}^{s_2}.\end{aligned} \tag{11}$$

From the previous section and Sobolev estimates, we claim the following:

Proposition 2.4 (Control of the space-time L^{p+1} norm of w) *For all $a \in \mathbb{R}^N$ and $s \geq -\log T + 1$,*

$$\int_s^{s+1} \int_B |w|^{p+1} \rho dy ds \leq C(C_0, N, p).$$

Proof: For $s \geq -\log T + 1$, let us work with time integrals between s_1 and s_2 where $s_1 \in [s-1, s]$ and $s_2 \in [s+1, s+2]$. We will first control all the terms on the right hand side of the relation (11) in terms of the space-time L^{p+1} norm of w. Hence, we conclude the estimate. In the following, C denotes a constant that depends only on p, N and C_0, and ϵ is an arbitrary positive number in $(0,1)$.

Step 1: Control of the H^1 norm of w in terms of its L^{p+1} norm
We claim the following:

Lemma 2.5

$$\int_{s_1}^{s_2} \int_B |\nabla w|^2 (1-|y|^2)^{\alpha+1} dy ds \leq C + \frac{2}{p+1} \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho dy ds, \quad (12)$$

$$\sup_{s_1 \leq s \leq s_2} \int_B w(y,s)^2 \rho dy \leq \frac{C}{\epsilon} + C\epsilon \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho dy ds. \quad (13)$$

Proof: Since $|y.\nabla w| \leq |y|.|\nabla w|$, it follows that

$$\int_B |\nabla w|^2 (1-|y|^2)^{\alpha+1} dy \leq \int_B \left(|\nabla w|^2 - (y.\nabla w)^2\right) \rho dy. \quad (14)$$

Using the energy integral (9) and the energy bound (7), we get (12).

By the mean value theorem, there exists $\tau \in [s_1, s_2]$ such that

$$\int_B w(y,\tau)^2 \rho dy = \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_B w^2 \rho dy ds \leq \int_{s_1}^{s_2} \int_B w^2 \rho dy ds \quad (15)$$

because $s_2 - s_1 \geq 1$. For any $s \in [s_1, s_2]$,

$$\begin{aligned} \int_B w(y,s)^2 \rho dy &= \int_B w(y,\tau)^2 \rho dy + \int_\tau^s \frac{d}{ds} \int_B w^2 \rho dy \\ &\leq \int_B w(y,\tau)^2 \rho dy + 2 \int_{s_1}^{s_2} \int_B |w||w_s| \rho dy ds. \end{aligned}$$

Using the fact that $2ab \leq a^2 + b^2$, we write

$$2 \int_{s_1}^{s_2} \int_B |w||w_s| \rho dy ds \leq \int_{s_1}^{s_2} \int_B w_s^2 \rho dy ds + \int_{s_1}^{s_2} \int_B w^2 \rho dy ds.$$

Using the bound on w_s (8), we get for all $s \in [s_1, s_2]$,

$$\int_B w(y,s)^2 \rho dy \leq C + C \int_{s_1}^{s_2} \int_B w^2 \rho dy ds.$$

Since $1 \le s_2 - s_2 \le 3$, we use Jensen's inequality to write

$$\int_{s_1}^{s_2}\int_B w^2\rho dyds \le C\left(\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds\right)^{\frac{2}{p+1}} \le \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds. \quad (16)$$

The desired bound (13) follows then from estimates (15) through (16). This concludes the proof of Lemma 2.5. ■

Step 2 : Control of the terms on the right hand side of the relation (11)
In this step, we prove the following identity

$$\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds \le C + C\int_B \left(w_s(y,s_1)^2 + w_s(y,s_2)^2\right)\rho dy. \quad (17)$$

For this, we will bound each term on the right hand side of (11) with the L^{p+1} norm. Note that the first term is bounded because of the energy bound (7), while the second is negative.

a) Control of $\int_{s_1}^{s_2}\int_B w_s y.\nabla w\rho dyds$:
Using the definition of ρ (5) and the Cauchy-Schwarz inequality, we write

$$\begin{aligned}\left|\int_{s_1}^{s_2}\int_B w_s y.\nabla w\rho dyds\right| &\le \int_{s_1}^{s_2}\int_B |w_s|(1-|y|^2)^{\frac{\alpha-1}{2}}|\nabla w|(1-|y|^2)^{\frac{\alpha+1}{2}}dyds\\ &\le \left(\int_{s_1}^{s_2}\int_B w_s^2(1-|y|^2)^{\alpha-1}dyds\right)^{1/2}\left(\int_{s_1}^{s_2}\int_B |\nabla w|^2(1-|y|^2)^{\alpha+1}dyds\right)^{1/2}\\ &\le \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B |w|^{p+1}dyds. \quad (18)\end{aligned}$$

where we used the bound on w_s (8) and the bound on the gradient (12).

b) Control of $\int_{s_1}^{s_2}\int_B w_s w y.\nabla\rho dyds$:
Since we have from the definition of ρ (5)

$$y.\nabla\rho = -2\alpha|y|^2(1-|y|^2)^{\alpha-1}, \quad (19)$$

we can use the Cauchy-Schwarz inequality to write

$$\begin{aligned}\left|\int_{s_1}^{s_2}\int_B w_s w y.\nabla\rho dyds\right| &\le 2\alpha\int_{s_1}^{s_2}\int_B |w_s|(1-|y|^2)^{\frac{\alpha-1}{2}}|w||y|(1-|y|^2)^{\frac{\alpha-1}{2}}dyds\\ &\le 2\alpha\left(\int_{s_1}^{s_2}\int_B w_s^2(1-|y|^2)^{\alpha-1}dyds\right)^{1/2}\left(\int_{s_1}^{s_2}\int_B w^2|y|^2(1-|y|^2)^{\alpha-1}dyds\right)^{1/2}\\ &\le \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B w^2|y|^2(1-|y|^2)^{\alpha-1}dyds,\end{aligned}$$

where we used the bound on w_s (8). Since we have the following Hardy type inequality for any $f \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^{\mathrm{N}})$ (see Appendix B for details):

$$\int_B f^2|y|^2(1-|y|^2)^{\alpha-1}dy \le C\int_B |\nabla f|^2(1-|y|^2)^{\alpha+1}dy + C\int_B f^2\rho dy, \quad (20)$$

we use the bound on the gradient (12) and Jensen's inequality (16) to write

$$\left|\int_{s_1}^{s_2}\int_B w_s w y.\nabla\rho dyds\right| \le \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds. \tag{21}$$

c) Control of $\int_B w w_s \rho dy$:
Using the fact that $ab \le a^2+b^2$ and the control (13) of the L^2 norm, we write

$$\begin{aligned}\left|\int_B w w_s \rho dy\right| &\le \int_B w_s^2\rho dy + \int_B w^2\rho dy\\ &\le \int_B w_s^2\rho dy + \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds.\end{aligned} \tag{22}$$

Now, we are able to conclude the proof of the identity (17) from the relation (11). For this, we bound all the terms on the right hand side of (11) (the second term is negative, use (7), (18), (21), (22) and (13) for the other terms) to get:

$$\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds \le \frac{C}{\epsilon} + C\epsilon\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds + C\int_B \left(w_s(y,s_1)^2 + w_s(y,s_2)^2\right)\rho dy.$$

Taking $\epsilon = 1/2C$ yields identity (17). ■

Step 3 : Conclusion of the proof

Let $s \ge -\log T + 1$. Using the mean value theorem, we get $s_1 \in [s-1, s]$ and $s_2 \in [s+1, s+2]$ such that

$$\int_{s-1}^{s}\int_B w_s(y,s)^2(1-|y|^2)^{\alpha-1}dyds = \int_B w_s(y,s_1)^2(1-|y|^2)^{\alpha-1}dy$$

and

$$\int_{s+1}^{s+2}\int_B w_s(y,s)^2(1-|y|^2)^{\alpha-1}dyds = \int_B w_s(y,s_2)^2(1-|y|^2)^{\alpha-1}dy.$$

Since the left hand sides of these inequalities are bounded by the bound on the w_s (8), it follows that

$$\int_B \left(w_s(y,s_1)^2 + w_s(y,s_2)^2\right)(1-|y|^2)^{\alpha-1}dy \le C.$$

Using the bound on the L^{p+1} norm of (17), we conclude that

$$\int_{s_1}^{s_2}\int_B |w|^{p+1}\rho dyds \le C.$$

Since $s_1 \le s \le s+1 \le s_2$, this concludes the proof of Proposition 2.4. ■

As a consequence of Proposition 2.4, estimate (8), Step 1 and the fact that $\frac{3}{4} \le 1-|y|^2 \le 1$ whenever $|y| \le \frac{1}{2}$, we have the following:

Corollary 2.6 (Bound on space-time norms of the solution) *For all $a \in \mathbb{R}^N$ and $s \geq -\log T + 1$, the following identities hold*

$$i) \quad \int_s^{s+1} \int_B \left(|w_a|^{p+1}\rho + \partial_s w_a(y,s)^2(1-|y|^2)^{\alpha-1} + |\nabla w_a|^2(1-|y|^2)^{\alpha+1}\right) dyds \leq C,$$

$$\int_B w_a(y,s)^2 \rho dy \leq C,$$

$$ii) \quad \int_s^{s+1} \int_{B_{1/2}} \left(\partial_s w_a(y,s)^2 + |\nabla w_a|^2 + |w_a|^{p+1}\right) dyds \leq C,$$

$$\int_{B_{1/2}} w_a^2 dy \leq C. \tag{23}$$

where $B_{1/2} \equiv B(0,1/2)$, $C = C(C_0, N, p)$ and C_0 depends only on a bound on T and the norm of initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

3 Control of the $\mathrm{H}^1_{\mathrm{loc,u}}$ norm of the solution

In this section, we conclude the proof of Theorem 1. Let us remark that Theorem 1' follows from Theorem 1 and the change of variables (3) as in [2]. We proceed in two steps:

- In the first step, we use the uniform local bounds we obtained in the previous section to gain more regularity on the solution by interpolation (control of the $\mathrm{L}^r_{\mathrm{loc}}$ norm of the solution, where $r \leq \frac{p+3}{2}$).

- In the second step, we use Gagliardo-Nirenberg type argument involving the functional E to conclude the proof.

Step 1: Control of $w_a(s)$ in $\mathrm{L}^r_{\mathrm{loc}}$

Proposition 3.1 *For all $s \geq -\log T + 1$ and $a \in \mathbb{R}^N$,*

$$\int_B |w_a(y,s)|^{\frac{p+3}{2}} dy \leq C \text{ if } N \geq 2 \text{ and } \int_B |w_a(y,s)|^{p+1} dy \leq C \text{ if } N = 1, \tag{24}$$

where B is the unit ball of $\mathbb{R}^N$.

Proof: We introduce $r = \frac{p+3}{2}$ for all $N \geq 2$ and $r = p+1$ for $N = 1$.

Let us first remark that thanks to a simple covering property, it is enough to prove the result with $B_{1/2}$ instead of B. Indeed, let us assume that

$$\text{for all } s \geq -\log T + 1 \text{ and } b \in \mathbb{R}^N, \quad \int_{B_{1/2}} |w_b(y,s)|^r dy \leq C \tag{25}$$

and prove (24). Consider $a \in \mathbb{R}^N$ and $s \geq -\log T + 1$. Remark that the ball B can be covered by a finite number $k(N)$ of balls of radius $\frac{1}{2}$. Thus, the problem reduces to controlling uniformly for $|y_0| < 1$,

$$\int_{|z-y_0|<\frac{1}{2}} |w_a(z,s)|^r dz.$$

Note that using the definition (3) of w_a, we see that

$$\text{for all } y \in \mathbb{R}^N, \;\; w_a(y+y_0,s) = w_{a+y_0e^{-s}}(y,s).$$

Therefore,

$$\int_{|z-y_0|<\frac{1}{2}} |w_a(z,s)|^r dz = \int_{|y|<\frac{1}{2}} |w_a(y+y_0,s)|^r dy = \int_{|y|<\frac{1}{2}} |w_{a+y_0e^{-s}}(y,s)|^r dy \leq C.$$

Let us prove (25) now. We write w for w_b.

i) Using Corollary 2.6 and the mean value theorem, we derive the existence of $\tau(s) \in [s, s+1]$ such that

$$\int_{B_{1/2}} |w(y,\tau)|^{p+1} dy = \int_s^{s+1} \left(\int_{B_{1/2}} |w|^{p+1} dy \right) ds \leq C.$$

Therefore, since $r \in [2, p+1]$, we use the Cauchy-Schwarz inequality and the L^2 bound in (23) to obtain

$$\int_{B_{1/2}} |w(y,\tau(s))|^r dy \leq C.$$

ii) Moreover, using again Corollary 2.6, and the fact that $ab \leq a^2 + b^2$, we write

$$\begin{aligned}
\int_{B_{1/2}} |w(y,s)|^r dy &= \int_{B_{1/2}} |w(y,\tau)|^r dy + \int_\tau^s \frac{d}{ds} \int_{B_{1/2}} |w|^r dyds \\
&\leq C + r \int_s^{s+1} \int_{B_{1/2}} |w_s||w|^{r-1} dyds \\
&\leq C + r \left(\int_s^{s+1} \int_{B_{1/2}} w_s^2 dyds + \int_s^{s+1} \int_{B_{1/2}} |w|^{2(r-1)} dyds \right) \\
&\leq C + r \int_s^{s+1} \int_{B_{1/2}} |w|^{2(r-1)} dyds.
\end{aligned}$$

In the case $r = \frac{p+3}{2}$, we have $2(r-1) = p+1$, hence, the last line is uniformly bounded by Corollary 2.6.
In the case $N = 1$, we have $r = p+1$ and $2(r-1) = 2p$. Using Sobolev's embedding in two dimensions (space and time), and Corollary 2.6, we write

$$\int_s^{s+1} \int_{B_{1/2}} |w|^{2p} dyds \leq C \left(\int_s^{s+1} \int_{B_{1/2}} \left(\partial_s w_a(y,s)^2 + |\partial_y w_a|^2 + w_a^2\right) dyds \right)^p \leq C.$$

This concludes the proof of Proposition 3.1. ■

Step 2: Control of the gradient in $\mathrm{L}^2_{\mathrm{loc,u}}$

We claim the following

Proposition 3.2 (Uniform control of the $\mathrm{H}^1_{\mathrm{loc,u}}$ norm of $w_a(s)$) *For all $s \geq -\log T + 1$ and $a \in \mathbb{R}^N$,*

$$\int_{B_{1/2}} |\nabla w_a(y,s)|^2 dy \leq C.$$

We first introduce the following estimate.

Lemma 3.3 (Local control of the space L^{p+1} norm by the H^1 norm) *For all $s \geq -\log T + 1$ and $a \in \mathbb{R}^N$,*

$$\int_B |w_a|^{p+1} \leq C + C\left(\int_B |\nabla w_a|^2 dy\right)^\beta,$$

where $\beta = \beta(p,N) \in [0,1)$.

Proof: If $N = 1$, Proposition 3.1 implies the result with $\beta = 0$. Assume now that $N \geq 2$. Since $1 < p < 1 + \frac{4}{N-1}$, it follows that $p+1 < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$. Therefore, we can introduce some $q = q(p,N)$ to be fixed later such that

$$\frac{p+3}{2} < p+1 \leq q \leq 2^*.$$

We have by interpolation and Proposition 3.1,

$$\int_B |w_a|^{p+1} \leq \left(\int_B |w_a|^{\frac{p+3}{2}}\right)^{1-\theta} \left(\int_B |w_a|^q\right)^\theta \leq C\left(\int_B |w_a|^q\right)^\theta,$$

where

$$\theta = \left(p+1-\frac{p+3}{2}\right) / \left(q - \frac{p+3}{2}\right) = \frac{p-1}{2q-(p+3)}.$$

Sobolev's embedding in the unit ball B, the fact that $q > \frac{p+3}{2}$ and Proposition 3.1 yield

$$\int_B |w_a|^{p+1} \leq C\left(\int_B |\nabla w_a|^2\right)^\beta + C\left(\int_B |w_a|^{\frac{p+3}{2}}\right)^{\frac{2\theta q}{p+3}} \leq \left(\int_B |\nabla w_a|^2\right)^\beta + C,$$

where

$$\beta(q) = \frac{q\theta}{2} = \frac{(p-1)q/4}{q - 3/2 - p/2}. \tag{26}$$

If $N \geq 3$, then we fix $q = 2^*$. Since $p < 1 + \frac{4}{N-1}$, it follows that

$$\beta = \frac{(p-1)2^*/4}{2^* - (p+3)/2} < \frac{2^*/(N-1)}{2^* - 3/2 - \frac{1}{2}(1 + \frac{4}{N-1})} = \frac{2^*}{(N-1)\left(\frac{4}{N-2} - \frac{2}{N-1}\right)} = 1.$$

If $N = 2$, just note from (26) that when $q \to \infty$, we have $\beta(q) \to \frac{p-1}{4} < 1$, because $1 < p < 1 + \frac{4}{N-1} = 5$. Therefore, we can fix q large enough such that $\beta(q) < 1$. This concludes the proof of Lemma 3.3. ■

Let us prove Proposition 3.2 now.

Proof of Proposition 3.2: We will prove that for some $C = C(N, p, C_0)$, we have

$$\text{for all } s \geq -\log T + 1 \text{ and } a \in \mathbb{R}^N, \quad \int_{B_{1/2}} |\nabla w_a(y,s)|^2 dy \leq C. \tag{27}$$

For a given $s \geq -\log T + 1$, there exists $a_0 = a_0(s)$ such that

$$\int_B |\nabla w_{a_0}|^2 (1-|y|^2)^{\alpha+1} dy \geq \frac{1}{2} \sup_{a \in \mathbb{R}^N} \int_B |\nabla w_a|^2 (1-|y|^2)^{\alpha+1} dy. \tag{28}$$

i) We claim that a covering argument and the definition of $a_0(s)$ yield

$$\int_B |\nabla w_{a_0}|^2 dy \leq C \int_B |\nabla w_{a_0}|^2 (1-|y|^2)^{\alpha+1} dy. \tag{29}$$

Indeed, since we can cover B with $k(N)$ balls of radius $1/2$, it is enough to prove that

$$\int_{|y|<\frac{1}{2}} |\nabla w_{a_0}(y+y_0, s)|^2 dy \leq C \int_B |\nabla w_{a_0}|^2 (1-|y|^2)^{\alpha+1} dy \tag{30}$$

uniformly for $|y_0| \leq 1$. Using the definition (3) of w, we see that

$$\text{for all } y \in \mathbb{R}^N, \nabla w_{a_0}(y+y_0, s) = \nabla w_{a_0+y_0e^{-s}}(y,s).$$

Therefore, since $1 - |y|^2 \geq \frac{3}{4}$ whenever $|y| \leq \frac{1}{2}$, we write

$$\begin{aligned}
&\int_{|y|<\frac{1}{2}} |\nabla w_{a_0}(y+y_0, s)|^2 dy = \int_{|y|<\frac{1}{2}} |\nabla w_{a_0+y_0e^{-s}}(y,s)|^2 dy \\
&\leq C \int_B |\nabla w_{a_0+y_0e^{-s}}(y,s)|^2 (1-|y|^2)^{\alpha+1} dy \leq C \sup_{a \in \mathbb{R}^N} \int_B |\nabla w_a|^2 (1-|y|^2)^{\alpha+1} dy \\
&\leq C \int_B |\nabla w_{a_0}|^2 (1-|y|^2)^{\alpha+1} dy,
\end{aligned}$$

by definition of the supremum (28). This yields (30) and then (29).

ii) From the estimates on the Lyapunov functional E and Lemma 3.3, we have the conclusion. Indeed, using the definition (6) of E, inequality (14) and the fact that $\alpha > 0$, we see that

$$\begin{aligned}
\int_B |\nabla w_{a_0}|^2 (1-|y|^2)^{\alpha+1} dy \quad &\leq \quad \int_B \left(|\nabla w_{a_0}|^2 - (y.\nabla w_{a_0})^2 \right) \rho dy \\
&= \quad 2E(w_{a_0}) + 2 \int_B \left(-\frac{1}{2} \partial_s w_{a_0}^2 - \frac{(p+1)}{(p-1)^2} w_{a_0}^2 + \frac{1}{p+1} |w_{a_0}|^{p+1} \right) \rho dy \\
&\leq \quad 2E(w_{a_0}) + \frac{2}{p+1} \int_B |w_{a_0}|^{p+1} dy.
\end{aligned}$$

Using the bound (7) on E, the control of the L^{p+1} by the H^1 norm of Lemma 3.3 and (29), we obtain

$$\int_B |\nabla w_{a_0}|^2(1-|y|^2)^{\alpha+1}dy \le C + C\left(\int_B |\nabla w_{a_0}|^2(1-|y|^2)^{\alpha+1}dy\right)^\beta,$$

where $\beta \in [0,1)$. Therefore, for some $C = C(p,N,C_0)$ independent of s, we have

$$\int_B |\nabla w_{a_0(s)}(y,s)|^2(1-|y|^2)^{\alpha+1}dy \le C.$$

From the definition of $a_0(s)$, this yields

$$\text{for all } s \ge -\log T + 1 \text{ and } a \in \mathbb{R}^N,\ \int_B |\nabla w_a(y,s)|^2(1-|y|^2)^{\alpha+1}dy \le C.$$

Since $1-|y|^2 \ge \frac{3}{4}$ whenever $|y| \le \frac{1}{2}$, the estimate (27) follows. This concludes the proof of Proposition 3.2. ∎

Step 3: Conclusion of the proof of Theorem 1

We conclude the proof of Theorem 1 here.

i) Uniform control of the $\mathrm{H}^1(B)$ *norm of* $w_a(s)$:

From Proposition 3.2 and by covering the unit ball B by $k(N)$ balls of radius $\frac{1}{2}$, we obtain

$$\text{for all } s \ge -\log T \text{ and } a \in \mathbb{R}^N,\ \int_B |\nabla w_a|^2 dy \le C.$$

Since $2 < p+1$, we use this bound and Lemma 3.3 to get for all $s \ge -\log T$ and $a \in \mathbb{R}^N$,

$$\left(\int_B w_a^2 dy\right)^{\frac{p+1}{2}} \le C\int_B w_a^{p+1}dy \le C + C\left(\int_B |\nabla w_a|^2 dy\right)^\beta \le C.$$

Thus,

$$\text{for all } s \ge -\log T \text{ and } a \in \mathbb{R}^N,\ \|w_a(s)\|_{\mathrm{H}^1(B)} \le C(N,p,C_0).$$

ii) Uniform control of the $\mathrm{L}^2(B)$ *norm of* $\partial_s w_a(s)$:

From the expression and the boundedness of E (see (7)), Part i) yields for all $s \ge -\log T + 1$ and $a \in \mathbb{R}^N$,

$$\begin{aligned}\int_{B_{1/2}} \partial_s w_a^2 dy &\le C\int_B \partial_s w_a^2 \rho dy\\ &\le 2CE(w) + 2C\int_B\left(-\frac{(p+1)}{(p-1)^2}w_a^2 + \frac{1}{p+1}|w_a|^{p+1}\right)\rho dy\\ &- C\int_B \left(|\nabla w_a|^2 - (y.\nabla w_a)^2\right)\rho dy \le C. \end{aligned} \quad (31)$$

From a covering argument, we conclude again that

$$\text{for all } s \geq -\log T \text{ and } a \in \mathbb{R}^N, \quad \|\partial_s w_a(s)\|_{\mathrm{L}^2(B)} \leq C(N,p,C_0). \tag{32}$$

Indeed, since the unit ball B can be covered by $k(N)$ balls of radius $\frac{1}{2}$, this reduces to prove that:

$$\text{for all } s \geq -\log T + 1, \;\; a \in \mathbb{R}^N \text{ and } |y_0| < 1, \;\; \int_{|y-y_0|<\frac{1}{2}} \partial_s w_a(y,s)^2 dy \leq C. \tag{33}$$

Consider $a \in \mathbb{R}^N$ and $|y_0| < \frac{1}{2}$. For all b and y in $\mathbb{R}^N$, $w_b(y,s) = w_a(y + (b-a)e^s, s)$ and

$$\partial_s w_b(y,s) = \partial_s w_a(y + (b-a)e^s, s) + (b-a)e^s.\nabla w_a(y + (b-a)e^s, s).$$

Taking $b = a + y_0 e^{-s}$, this gives

$$\text{for all } y \in \mathbb{R}^N, \;\; \partial_s w_a(y + y_0, s)^2 \leq 2\partial_s w_{a+y_0e^{-s}}(y,s)^2 + 2|\nabla w_a(y,s)|^2.$$

Therefore, using (31) and Part i), we obtain (33) and then (32). This concludes the proof of Theorem 1. ■

A Evolution of the L^2_ρ norm of solutions of (4)

We prove estimate (10) here. For simplicity, we write $\iint$ for $\int_{s_1}^{s_2}\int_B$ and drop down $dyds$. If we multiply equation (4) by $w\rho$ and integrate in space and time over $B \times (s_1, s_2)$, then we get:

$$\iint \left(|w|^{p+1} - \frac{2(p+1)}{(p-1)^2}w^2\right)\rho = \iint \left(w_{ss} + \frac{(p+3)}{p-1}w_s\right) w\rho + 2\iint y.\nabla w_s w\rho$$
$$- \iint w \,\mathrm{div}\,(\rho\nabla w - \rho(y.\nabla w)y) \tag{34}$$

Since $2w_s w = \partial_s(w^2)$, we integrate by parts in time and write

$$\iint \left(w_{ss} + \frac{(p+3)}{p-1}w_s\right) w\rho = \left[\int_B \left(w_s w + \frac{p+3}{2(p-1)}w^2\right)\rho dy\right]_{s_1}^{s_2} - \iint w_s^2\rho. \tag{35}$$

Integrating by parts in space, we write

$$\begin{aligned} 2\iint y.\nabla w_s w\rho &= -2\iint w_s \nabla.\,(yw\rho) \\ &= -2N\iint w_s w\rho - 2\iint w_s y.\nabla w\rho - 2\iint w_s w y.\nabla\rho \\ &= -N\left[\int_B w^2\rho dy\right]_{s_1}^{s_2} - 2\iint w_s y.\nabla w\rho - 2\iint w_s w y.\nabla\rho. \end{aligned} \tag{36}$$

Integrating by parts in space, we write

$$-\iint w \,\mathrm{div}\,(\rho\nabla w - \rho(y.\nabla w)y) = \iint \left(|\nabla w|^2 - (y.\nabla w)^2\right)\rho. \tag{37}$$

Using (35), (36) and (37), we see that (34) yields the desired identity (10). ■

B A Hardy type identity

We prove the identity (20) here: For any f such that the right hand side is finite:

$$\int_B f^2|y|^2(1-|y|^2)^{\alpha-1}dy \le C\int_B |\nabla f|^2(1-|y|^2)^{\alpha+1}dy + C\int_B f^2\rho dy. \quad (38)$$

Using the expression of $y.\nabla\rho$ (19), we see that

$$\int_B f^2|y|^2(1-|y|^2)^{\alpha-1}dy = -\frac{1}{2\alpha}\int_{s_1}^{s_2}\int_B f^2 y.\nabla\rho dy.$$

If we integrate by parts in space, then we see that

$$-\int_B f^2 y.\nabla\rho dy = 2\int_B f\nabla f.y\rho dy + N\int_B f^2\rho dy. \quad (39)$$

Therefore, using the Cauchy-Schwarz inequality, we write

$$\begin{aligned} \left|\int_B f\nabla f.y\rho dy\right| &\le \int_B |\nabla f|(1-|y|^2)^{\frac{\alpha+1}{2}}|f||y|(1-|y|^2)^{\frac{\alpha-1}{2}}dy \\ &\le \left(\int_B |\nabla f|^2(1-|y|^2)^{\alpha+1}dyds\right)^{\frac12}\left(\int_B f^2|y|^2(1-|y|^2)^{\alpha-1}dy\right)^{\frac12} \\ &\le \frac{1}{\epsilon}\int_B |\nabla f|^2(1-|y|^2)^{\alpha+1}dy + \epsilon\int_B f^2|y|^2(1-|y|^2)^{\alpha-1}dy \end{aligned}$$

for any $\epsilon > 0$. Taking $\epsilon = \frac{\alpha}{5}$, we get the desired conclusion (38). ■

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