THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE CERGY-PONTOISE

spécialité

MATHÉMATIQUES

présentée par

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Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE CERGY-PONTOISE

Sujet de la thèse:

SUR LA DESCRIPTION DES FORMATIONS DE SINGULARITÉS POUR L'ÉQUATION DE LA CHALEUR NON LINÉAIRE

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REMERCIEMENTS

Je tiens à exprimer ma plus profonde gratitude à Frank Merle qui a dirigé mes travaux avec beaucoup de disponibilité et de bienveillance. Travailler avec lui fût pour moi un grand honneur et une occasion de profiter de son esprit vif et de sa grande culture scientifique.

Je remercie vivement Jean Bricmont qui a accepté d'être rapporteur sur ma thèse. Plusieurs de ses travaux m'ont été une source d'inspiration.
Mes remerciements vont aussi à Patrick Gérard qui a bien voulu rapporter sur ma thèse, et qui m'a constamment apporté son soutien.

Je remercie également Thierry Cazenave pour avoir accepté d'être parmi le jury de ma thèse.

Ma plus vive reconnaissance va à Henri Berestycki et Yann Brenier qui ont bien voulu faire partie de mon jury de thèse. Je tiens à les remercier aussi pour les encouragements qu'ils n'ont cessé de m'apporter tout au long de ma scolarité.

Abbas Bahri me fait l'honneur d'être dans le jury, et je l'en remercie très chaleureusement. Je lui exprime aussi ma reconnaissance pour son invitation à Rutgers University en décembre 97.

Cette thèse n'aurait pas pu être possible sans l'excellent environnement scientifique et humain du Département de Mathématiques et d'Informatique de l'École Normale Supérieure et du Département de Mathématique de l'Université de Cergy-Pontoise.

Enfin, un grand merci à toute ma famille, à mes amis et à tous ceux qui m'ont aidé un jour.

À la mémoire de mon père Aboubaker
et de mon grand-père Salem,
à ma mère Zahia, à mon frère Mohamed Lotfi.

Résumé

On s'intéresse au phénomène d'explosion en temps fini dans les équations du type:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \tag{1}$$

où $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $1 < p$, $(N-2)p < N+2$.

Dans une première direction, on construit pour (1) une solution u qui explose en temps fini $T > 0$ en un seul point d'explosion $x_0 \in \mathbb{R}^N$, et on décrit complètement le profil (ou comportement asymptotique) de u à l'explosion. Cette construction s'appuie sur la technique d'estimations a priori des solutions explosives de (1) qui permet une réduction en dimension finie du problème, et sur un lemme de type Brouwer. La méthode utilisée permet de dégager un résultat de *stabilité* du comportement de la solution construite par rapport à des perturbations dans les données initiales ou dans le terme non linéaire de réaction. De plus, la méthode se généralise à des équations vectorielles de type chaleur avec non-linéarité sans structure de gradient, ainsi qu'au traitement d'un problème de reconnexion d'un vortex avec la paroi en supra-conductivité.

Dans une seconde direction, on s'intéresse à l'équation suivante associée à (1):

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - \frac{w}{p-1} + w^p, \tag{2}$$

et on démontre un Théorème de Liouville qui donne une classification des solutions de (2) globales en temps et en espace et uniformément bornées. On obtient également une propriété de localisation de l'équation (1) (si $u \geq 0$) qui permet de la comparer de façon précise à la solution de l'équation différentielle associée.

Enfin, on s'intéresse de nouveau à la notion de profil et on utilise les estimations qui découlent du Théorème de Liouville pour prouver un résultat d'équivalence de différentes notions de profils d'explosion ou de développement asymptotique de u au voisinage de x_0 point d'explosion, en variable x, $y = \frac{x-x_0}{\sqrt{T-t}}$ ou $z = \frac{x-x_0}{\sqrt{(T-t)|\log(T-t)|}}$.

Mots clés: équation de la chaleur, singularité, explosion en temps fini, extinction en temps fini, profil, développement asymptotique, équations vectorielles, supra-conductivité.

Abstract

We are interested in finite-time blow-up phenomena for heat equations of the type:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \tag{1}$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $1 < p$, $(N-2)p < N+2$.

We first construct for (1) a solution u which blows-up in finite time T at only one blow-up point $x_0 \in \mathbb{R}^N$, and describe completely its blow-up profile (or asymptotic behavior). This construction is based on *a priori* estimates' technique which reduces the problem to a finite-dimensional one, and on a Brouwer type lemma. This method allows us to derive a stability result of the behavior of u with respect to initial data or perturbation of the nonlinearity. In addition, we generalize the method to the case of vector-valued equations with a non gradient nonlinearity, as well as a vortex reconnection with the boundary in super-conductivity.

In a second step, we consider the following equation derived form (1):

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - \frac{w}{p-1} + w^p, \tag{2}$$

and prove a Liouville Theorem which classifies all uniformly bounded globally (in space and time) defined solutions of (2). We then obtain a localization property of equation (1) (if $u \geq 0$) which allows a precise comparison with solutions of the associated ordinary differential equation.

In a third step, we use a consequence of the Liouville Theorem to prove the equivalence of different notions of blow-up profile or asymptotic behavior near a blow-up point x_0 of u, namely in variables x, $y = \frac{x-x_0}{\sqrt{T-t}}$ or $z = \frac{x-x_0}{\sqrt{(T-t)|\log(T-t)|}}$.

Key words: heat equation, singularity, finite time blow-up, finite time quenching, profile, asymptotic behavior, vector-valued equations, super-conductivity.

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Chapitre 1

Introduction

L'objet de cette thèse est l'étude de la formation en temps fini de singularités dans des systèmes de réaction-diffusion de type chaleur:

$$(1)\qquad \begin{cases} \frac{\partial u}{\partial t} &= \Delta u + F(u) & \text{dans} & \Omega \times [0,T) \\ u &= 0 & \text{sur} & \partial\Omega \times [0,T) \\ u(.,0) &= u_0 & \text{dans} & \Omega \end{cases}$$

où
$u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}^M$, $u_0 : \Omega \to \mathbb{R}^M$,
Ω est un ouvert convexe borné et régulier de $\mathbb{R}^N$ ou $\Omega = \mathbb{R}^N$, $T > 0$,
$(\Delta u)_i = \Delta u_i$,
$F : \mathbb{R}^M \to \mathbb{R}^M$ est de classe C^1,
et $N, M \in \mathbb{N}$. (La condition de bord est à ignorer si $\Omega = \mathbb{R}^N$).

Ce système constitue un modèle simplifié pour beaucoup de phénomènes physiques de réaction-diffusion. Il apparaît notamment en combustion (voir Williams [58], Kapila [31], Kassoy et Poland [33], [34] (explosions thermiques), Bebernes et Eberly [2] (en particulier, un modèle de combustion solide), Bebernes et Kassoy [3], Lacey [36], Galaktionov, Kurdyumov et Samarskii [19], Galaktionov et Vazquez [20]). On le retrouve aussi dans beaucoup de situations physiques, de la mécanique des fluides à l'optique, sous la forme de l'équation de Ginzburg-Landau complexe (voir Levermore et Oliver [37]). Le système (1) a également un intérêt en neuro-biologie (voir Nagasawa [49], McKean [41]), et dans des modèles génétiques (voir Fisher [14]).

Le problème de Cauchy (local en temps) pour (1) peut être résolu dans une grande classe d'espaces fonctionnels. Citons par exemple l'espace des fonctions de $C(\bar{\Omega})$ nulles sur $\partial\Omega$ (si Ω est borné) ainsi que l'espace $H_0^1 \cap L^\infty(\Omega)$ que nous considérons dans la suite (voir Friedman [15], Henry [27], Pazy [50], Weissler [56]).
On peut alors définir $T > 0$ comme étant le temps maximum d'existence de la solution de (1). D'après la théorie locale, $u \in C\left([0,T), H_0^1 \cap L^\infty(\Omega)\right)$.
Deux cas se présentent:
- $T = +\infty$: existence globale.
- $T < +\infty$: dans ce cas,

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T} \|u(t)\|_{H^1(\Omega)} = +\infty.$$

On dit alors que u explose en temps fini T.

Par la suite, on s'intéresse à l'étude de telles solutions explosives.
Pour cela, on introduit la notion de point d'explosion (voir par exemple Friedman et McLeod [17]):

Définition 1 *Un point $a \in \Omega$ est dit point d'explosion de u si $u(x,t)$ n'est pas localement bornée au voisinage de (a,T), autrement dit s'il existe $(x_n,t_n) \to (a,T)$ tel que $|u(x_n,t_n)| \to +\infty$ quand $n \to +\infty$.*

Il est classique que

Si u explose en temps fini T, alors elle admet (au moins) un point d'explosion.

Plusieurs questions se posent autour de l'étude de l'explosion en temps fini dans (1):

Question 1: Existence. Existe-t-il des solutions de (1) qui explosent en temps fini? Existe-t-il des conditions suffisantes sur u_0 et F qui entraînent l'explosion?

Question 2: Normes de u. Peut-on avoir des estimations précises des normes (spatiales) de u et de ses dérivées à l'explosion?

Question 3: Comportement asymptotique. Comment se comporte u asymptotiquement au voisinage de (a, T) où $a \in \Omega$ est un point d'explosion? Est-ce que ce comportement est universel (i.e. indépendant des données initiales)? Est-il stable par rapport à des perturbations dans les données initiales ou le terme non linéaire?

Question 4: Interaction de la diffusion et de la réaction. Peut-on comparer l'équation (1) à une équation de dynamique locale en espace (une équation différentielle), ce qui permettrait de comprendre les rôles respectifs et les interactions entre le terme de diffusion Δu et le terme de réaction $F(u)$, surtout au voisinage de l'explosion.

1 Aperçu historique et directions fondamentales de l'étude

Dans la littérature sur l'explosion en temps fini pour le système (1), le cas suivant a constitué un prototype intéressant retenu par plusieurs auteurs:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \tag{2}$$

avec $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}$, Ω ouvert de $\mathbb{R}^N$, $1 < p$ et si $N \geq 3$, $p < \frac{N+2}{N-2}$. Dans le cas $N = M = 1$, d'autres auteurs se sont intéressés au cas de

$$\frac{\partial u}{\partial t} = \Delta u + e^u. \tag{3}$$

Nous nous intéressons essentiellement à l'équation prototype (2).

Les premières conditions suffisantes d'existence d'une solution explosive pour les équations de type (2) sont dues en particulier à Kaplan [32], Friedman [16], Fujita [18], Levine [38], Ball [1] et Weissler [57]. Par exemple, dans le cas d'un domaine borné Ω, Ball a obtenu, grâce à l'énergie associée à (2)

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx,$$

et à des méthodes d'équations différentielles ordinaires, une obstruction à l'existence globale d'une solution de (2):

Si $u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$ et $E(u_0) \leq 0$, alors $u(t)$ explose en temps fini.

Dans un contexte plus général ($\Omega = \mathbb{R}^N$ ou Ω borné), Giga et Kohn [23] se sont appuyés sur l'énergie locale pondérée suivante:

$$\begin{aligned} \mathcal{E}_{a,t}(\varphi) &= t^{\frac{2}{p-1}-\frac{N}{2}+1}\int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{p+1}|\varphi|^{p+1}\right) e^{-|x-a|^2/4t} dx \\ &+ t^{\frac{2}{p-1}-\frac{N}{2}}\int_\Omega \frac{1}{2(p-1)}|\varphi|^2 e^{-|x-a|^2/4t} dx \end{aligned} \tag{4}$$

pour trouver une condition nécessaire de non explosion au voisinage d'un point donné:

Il existe une constante $\sigma(N,p) > 0$ telle que si $u(t)$ explose en temps fini T et si Ω est étoilé en un point $a \in \Omega$ vérifiant

$$\mathcal{E}_{a,T}(u_0) < \sigma,$$

alors a n'est pas point d'explosion pour $u(t)$.

L'étude du comportement asymptotique de $u(t)$ au voisinage de (a,T) où a est un point d'explosion s'est faite d'abord grâce à l'introduction de variables auto-similaires:

$$y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),\ w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t). \tag{5}$$

D'après (2), w_a (ou simplement w) vérifie: $\forall s \geq -\log T$, $\forall y \in W_{a,s}$,

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w \tag{6}$$

où $W_{a,s} = (\Omega - a)e^{\frac{s}{2}}$.
Ainsi, l'étude de $u(t)$ au voisinage de (a,T) est équivalente à l'étude du comportement asymptotique de $w_a(s)$ quand $s \to +\infty$. D'ailleurs, l'énergie locale $\mathcal{E}_{a,t}$ définie en (4) n'est autre que l'équivalent pour u de la fonctionnelle de Liapunov associée à (6).

Giga et Kohn ont démontré dans [21] et [22] que si

$$u_0 \geq 0$$

ou

$$p < \frac{3N+8}{3N-4} \text{ ou } N = 1,$$

alors il existe $\epsilon_0 > 0$ et $C > 0$ tels que

$$0 < \epsilon_0 \leq \lim_{s\to+\infty} \|w(s)\|_{L^\infty(W_{a,s})} \leq \frac{1}{\epsilon_0} \tag{7}$$

et

$$\|\nabla w(s)\|_{L^\infty} + \|\Delta w(s)\|_{L^\infty} + \|\nabla \Delta w(s)\|_{L^\infty} \leq C. \tag{8}$$

Ceci revient à dire en terme de u que

$$\epsilon_0 \leq \lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty} \leq \frac{1}{\epsilon_0}$$

et

$$\begin{aligned} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty} + (T-t)^{\frac{1}{p-1}+1} \|\Delta u(t)\|_{L^\infty} & \\ + (T-t)^{\frac{1}{p-1}+\frac{3}{2}} \|\nabla \Delta u(t)\|_{L^\infty} & \leq C. \end{aligned}$$

Une première approche dans la recherche d'un développement asymptotique pour w_a a consisté en une étude de (6) dans L^2_ρ où

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}, \tag{9}$$

ce qui a permis d'avoir des convergences de $w(s)$ quand $s \to +\infty$ valables uniformément sur tout compact. Cette étude a été initiée par Giga et Kohn qui ont démontré dans [22] et [23] qu'il existe $l_a \in \{-\kappa, \kappa\}$ tel que

$$\sup_{|y|\le R} |w_a(y,s) - l_a| \to 0 \text{ quand } s \to +\infty$$

où

$$\kappa = (p-1)^{-\frac{1}{p-1}}.$$

Notons que κ, $-\kappa$ et 0 sont les seules solutions de (6) indépendantes du temps. Ce résultat a été précisé dans le cas $\Omega = \mathbb{R}^N$ par Filippas et Kohn [11], Filippas et Liu [13], et Herrero et Velázquez [28], [53] (voir aussi les articles de revue [30] et [52]), grâce à une analyse dans des espaces de Sobolev avec poids Gaussien (9). Ces auteurs ont montré que deux cas peuvent se produire:

- soit il existe $k \in \{0, ..., N-1\}$ et une matrice orthonormée Q tels que

$$(10)\quad \forall R > 0, \quad \sup_{|y|\le R} \left| w_a(y,s) - \kappa - \frac{\kappa}{2ps}\left((N-k) - \frac{1}{2} y^T A_k y\right)\right| = o\left(\frac{1}{s}\right)$$

où

$$A_k = Q^{-1} \begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix} Q$$

et I_{N-k} est la matrice identité $(N-k) \times (N-k)$,

- soit il existe $\delta > 0$ tel que

$$\forall R > 0, \quad \sup_{|y|\le R} |w_a(y,s) - \kappa| \le C(R)e^{-\delta s}.$$

Herrero et Velázquez ont affiné ce cas de convergence exponentielle à l'ordre 1 (voir [29] et [53]).

D'un point de vue physique, ces développements asymptotiques sont insuffisants. En effet, une fois traduite dans les variables d'origine (x,t), la convergence est uniforme uniquement à l'intérieur de paraboles du type $|x-a| \le R\sqrt{T-t}$, ce qui ne permet pas de déduire un profil asymptotique de $u(t)$ dans la variable x.

Néanmoins, le domaine de convergence a pu être étendu par Herrero et Velázquez [28], [55] (voir aussi [54]) aux ensembles $|z| \le C$ où

$$z = \frac{y}{\sqrt{s}}$$

en dimension N. Ils se sont appuyés sur une estimation linéaire dans des espaces de Sobolev avec poids Gaussien de l'effet du terme convectif $-\frac{1}{2}y.\nabla w$ dans l'équation (6). Ce résultat de Herrero et Velázquez leur a permis de dégager dans le cas (10) une notion de profil limite pour la fonction u au sens où $u(x,t) \to u^*(x)$ quand $t \to T$ si $x \ne a$ et x est voisin de a, avec

$$(11)\qquad u^*(x) \sim \left[\frac{8p|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1}{p-1}} \quad \text{quand } x \to a.$$

Ce problème a été également exploré d'un point de vue numérique. Citons en particulier une étude numérique de Berger et Kohn [4] qui a permis de dégager l'existence d'un profil asymptotique pour certaines solutions de (6)

$$f(z) = \left(p - 1 + \frac{(p-1)^2}{4p} |z|^2 \right)^{-\frac{1}{p-1}}. \tag{12}$$

Il a été observé numériquement dans [4] que

$$w(y,s) \sim f\left(\frac{y}{\sqrt{s}}\right) \text{ quand } s \to +\infty.$$

Bricmont et Kupiainen ont démontré dans [6] (voir aussi [5] et Bricmont, Kupiainen et Lin [7]) l'existence d'une donnée initiale pour w telle que

$$\sup_{y \in \mathbb{R}^N} \left| w(y,s) - f\left(\frac{y}{\sqrt{s}}\right) \right| \to 0 \text{ quand } s \to +\infty. \tag{13}$$

Grâce à la transformation (5), ceci donne pour tout $a \in \mathbb{R}^N$ une solution explosive $u(t)$ de (2) telle que

$$\sup_{x \in \mathbb{R}^N} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - f\left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right) \right| \to 0 \tag{14}$$

quand $t \to T$.

Nous nous proposons de développer trois directions dans cette thèse:

- Dans une première direction, on propose une seconde méthode de démonstration du résultat (13) de Bricmont et Kupiainen, basée sur la technique d'estimations a priori des solutions de (6) qui permet une réduction en dimension finie du problème, et sur un lemme de type Brouwer. Cette méthode permet de dégager un résultat de *stabilité* du comportement (13) par rapport à des perturbations dans les données initiales ou dans le terme non linéaire de réaction. D'autre part, la méthode se généralise à des équations vectorielles de type chaleur avec non-linéarité sans structure de gradient, ainsi qu'au traitement d'un problème de reconnexion d'un vortex avec la paroi en supra-conductivité.

- Dans une seconde direction, on affine les estimations (7) et (8) de Giga et Kohn, grâce à un Théorème de Liouville qui donne une classification des solutions globales de (6). On obtient également une propriété de localisation de l'équation (2) qui permet de la comparer de façon précise à la solution de l'équation différentielle associée.

- Enfin, on s'intéresse de nouveau à la notion de profil et on utilise les estimations qui découlent du Théorème de Liouville pour prouver un résultat d'équivalence des trois notions de profils d'explosion ou de développements asymptotiques en variable x, y, ou $z = \frac{y}{\sqrt{s}}$.

2 Existence et stabilité d'une solution de (2) avec les comportements (14) et (11)

a) Équation de la chaleur avec une non-linéarité en puissance

On considère le problème de construction d'une solution de l'équation

$$(15)\qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \\ u(x,t) \in \mathbb{R},\ x \in \mathbb{R}^N,\ t \geq 0 \\ 1 < p,\ \text{et si } N \geq 3,\ p < \frac{N+2}{N-2} \end{cases}$$

qui explose en temps fini donné $T > 0$ en un point unique donné $a \in \mathbb{R}^N$, et telle que u vérifie (14) et (11).

Dans [48] et [59] (voir aussi [47]), le résultat suivant a été obtenu (Théorème 1 page 48, Théorème 1 page 93 et Proposition 1 page 93):

Théorème 1 (Existence) *Il existe $T_0 > 0$ tel que pour tous $0 < \hat{T} \leq T_0$ et $\hat{a} \in \mathbb{R}^N$, il existe $\hat{u}_0 \in L^\infty \cap W^{1,p+1}(\mathbb{R}^N)$ telle que l'équation (15) avec donnée initiale $\hat{u}_0$ admet une solution $\hat{u}(t)$ explosant en temps fini $\hat{T}$ uniquement au point $\hat{a} \in \mathbb{R}^N$, et qui vérifie:*
i)

$$(16)\qquad \sup_{x\in\mathbb{R}^N} \left| (\hat{T}-t)^{\frac{1}{p-1}} \hat{u}(x,t) - f\left(\frac{x-\hat{a}}{\sqrt{(\hat{T}-t)|\log(\hat{T}-t)|}} \right) \right| \to 0$$

quand $t \to \hat{T}$ où f est définie en (12),
ii) $\forall x \in \mathbb{R}^N \backslash \{\hat{a}\}$, $\hat{u}(x,t) \to \hat{u}^*(x)$ quand $t \to \hat{T}$ et

$$(17)\qquad \hat{u}^*(x) \sim \left[\frac{8p|\log|x-\hat{a}||}{(p-1)^2|x-\hat{a}|^2} \right]^{\frac{1}{p-1}} \quad \textit{quand } x \to \hat{a}.$$

Signalons d'abord que *ii*) est une conséquence directe de *i*) grâce à l'invariance de l'équation (15) sous la transformation

$$\lambda \to u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t),$$

et à des estimations de régularité parabolique de (15) s'appuyant sur une condition suffisante de non explosion de solutions de (15) due à Giga et Kohn [23] (voir section 4 dans [59]).

L'objet du théorème se réduit donc à la construction d'une donnée initiale u_0 pour (15) telle que *i*) soit satisfaite. La preuve de ceci s'appuie sur:

1) La transformation du problème grâce à (5) et à des estimations a priori sur les solutions de (6) au voisinage du profil f défini en (12), ce qui permet de réduire le problème de construction à un problème de dimension finie,

2) Une résolution de ce problème de dimension finie à l'aide d'un argument topologique.

La méthode de réduction en dimension finie initiée pour la preuve du Théorème 1 dans [48] permet d'obtenir un résultat de stabilité du comportement (16) (et donc de (17)) par rapport à des perturbations $L^\infty \cap W^{1,p+1}(\mathbb{R}^N)$ de la donnée initiale. Plus précisément (Théorème 2 page 50 et Théorème 3 page 114):

Théorème 2 (Stabilité du comportement (16) et (17)) *Soit $\hat{u}_0$ la donnée initiale construite au Théorème 1. Soit $\hat{u}(t)$ la solution de (15) avec donnée initiale $\hat{u}_0$ qui explose en temps fini $\hat{T}$ en un point $\hat{a}$. Alors, pour tout $\epsilon > 0$, il existe $\mathcal{V}_\epsilon$ voisinage de $\hat{u}_0$ dans $L^\infty \cap W^{1,p+1}(\mathbb{R}^N)$ tel que pour* *tout* $u_0 \in \mathcal{V}_\epsilon$*, la solution* $u(t)$ *de (15) avec donnée initiale* u_0 *explose en temps fini* T *en un point unique* $a \in \mathbb{R}^N$ *tels que*

$$|a - \hat{a}| + |T - \hat{T}| \le \epsilon.$$

De plus, $u(t)$ *se comporte comme (16) et (17) avec* (a, T) *remplaçant* $(\hat{a}, \hat{T})$.

Remarque: Par les techniques de localisation présentées à la fin de la thèse, on peut avoir le même résultat dans l'espace d'énergie H^1.

La preuve du Théorème 2 s'appuie fondamentalement sur la technique de réduction en dimension finie du Théorème 1 ainsi que sur l'invariance de (6) sous l'action de la transformation géométrique

$$(a, T) \to w_{a,T}(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t)$$

où $y = \frac{x-a}{\sqrt{T-t}}$, $s = -\log(T - t)$, associée à une condition de non dégénérescence lorsque $u(x, t)$ est au voisinage du profil

$$(\hat{T} - t)^{-\frac{1}{p-1}} f\left(\frac{x - \hat{a}}{\sqrt{(\hat{T} - t)|\log(\hat{T} - t)|}}\right).$$

b) Équation de la chaleur complexe

La technique de réduction à un problème de dimension finie s'applique en fait dans un cadre beaucoup plus général que (15), celui des équations vectorielles avec une non-linéarité ne dérivant pas nécessairement d'un gradient (voir section 5 dans [59]). Un prototype d'une telle équation est le suivant:

$$(18) \qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + (1 + i\delta)|u|^{p-1}u \\ u(x, t) \in \mathbb{C},\ x \in \mathbb{R}^N,\ t \ge 0 \\ 1 < p,\ \text{et si } N \ge 3,\ p < \frac{N+2}{N-2}. \end{cases}$$

Dans [59], une solution explosive stable de (18) est construite dans le cas où δ est petit (voir Théorème 1 page 93 et Proposition 1 page 93):

Théorème 3 (Existence) *Il existe* $\delta_0 > 0$ *et* $T_0 > 0$ *tels que pour tous* $\delta \in [-\delta_0, \delta_0]$, $0 < T \le T_0$ *et* $a \in \mathbb{R}^N$*, il existe* $u_0 \in L^\infty \cap W^{1,p+1}(\mathbb{R}^N, \mathbb{C})$ *telle que l'équation (18) avec donnée initiale* u_0 *admet une solution* $u(t)$ *explosant en temps fini* T *uniquement au point* $a \in \mathbb{R}^N$ *et qui vérifie:*
i)

$$\sup_{x \in \mathbb{R}^N} \left| (T - t)^{\frac{1+i\delta}{p-1}} u(x, t) - f_\delta\left(\frac{x - a}{\sqrt{(T - t)|\log(T - t)|}}\right)^{1+i\delta} \right| \to 0$$

quand $t \to T$*, où*

$$f_\delta(z) = \left(p - 1 + \frac{(p-1)^2}{4(p - \delta^2)}|z|^2\right)^{-\frac{1}{p-1}},$$

ii) $\forall x \in \mathbb{R}^N \setminus \{a\}$, $u(x, t) \to u^*_\delta(x)$ *quand* $t \to T$ *et*

$$u^*_\delta(x) \sim \left[\frac{8(p - \delta^2)|\log|x - a||}{(p-1)^2|x - a|^2}\right]^{\frac{1+i\delta}{p-1}} \quad \text{quand } x \to a.$$

Bien que ce Théorème soit d'apparence très similaire au Théorème 1, il en diffère sur deux points:
1) Le Théorème 3 présente un comportement complètement complexe, au sens où le profil limite obtenu n'a pas de direction fixe dans $\mathbb{C}$.
2) La preuve du Théorème 3 qui s'appuie fondamentalement sur la technique de réduction en dimension finie introduite dans le cas réel, présente néanmoins une difficulté de plus sous la forme d'une direction dégénérée supplémentaire dans le problème. Cette difficulté est maîtrisée grâce à la théorie de la modulation (voir Filippas et Merle [12] pour un usage similaire de la théorie de la modulation). Le Théorème 2 de [59] généralise ce résultat au cas vectoriel (voir page 115).

c) Un problème d'extinction en temps fini

Comme autre application, le cas de l'équation (2) avec un terme d'amortissement de la non-linéarité $\nu\frac{|\nabla u|^2}{u}$ où $\nu \in (1,p)$ est traité de façon analogue dans [45] (voir Tayachi [51] où un terme d'amortissement de la forme $|\nabla u|^q$ est considéré). En effet, il est montré dans [45] que l'équation

$$(19) \qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nu\frac{|\nabla u|^2}{u} + N(u) \\ u(x,t) \in \mathbb{R},\ x \in \mathbb{R}^N,\ t > 0 \\ N(u) \sim u^p \text{ quand } u \to +\infty \\ |N(u)| \le C\exp(-\frac{1}{u}) \text{ si } |u| \le 1 \\ 1 < \nu < p, \end{cases}$$

admet une solution explosive en temps fini stable avec des comportements analogues à (16) et (17) (voir Proposition 1 page 133).
Remarque: Signalons que si $\nu \le 1 < p$ dans (19), alors des changements de fonctions évidents ramènent (19) au cas de (2) ou (3), deux cas où l'on dispose déjà de solutions explosives (voir la remarque après Proposition 1 page 133).

Grâce à une transformation du type

$$h(x,t) = \frac{1}{u(x,t)},$$

le résultat pour l'équation (19) donne un résultat d'extinction en temps fini pour le problème suivant:

$$(20) \qquad \begin{cases} \frac{\partial h}{\partial t} &= \Delta h - G(h) & \text{dans} & \Omega \times [0,T) \\ h(x,t) &= 1 & \text{sur} & \partial\Omega \times [0,T) \end{cases}$$

où Ω est un ouvert borné,

$$G(h) \sim \frac{1}{h^\beta} \text{ quand } h \to 0$$

et $\beta > 0$.

Si h est définie sur $\Omega \times [0,T)$ et

$$\lim_{t\to T} \inf_{x\in\Omega} h(x,t) = 0,$$

alors on dit que h s'éteint en temps fini.

L'équation (20) constitue un modèle de reconnexion d'un vortex avec la paroi dans un semi-conducteur de type II si $\beta = 1$ (voir Chapman, Hunton et

Ockendon [8]). Elle est également reliée à l'équation de diffusion générée par des phénomènes de polarisation dans des conducteurs ioniques (voir Kawarada [35]).

Quelques critères d'extinction en temps fini pour (20) étaient déjà connus en dimension 1 (voir Kawarada [35], Levine [39] (article de revue), [40]). Cependant, peu de choses étaient connues sur le comportement de la solution à l'extinction, sauf en ce qui concerne la localisation des points d'extinction (voir Guo [24], Deng et Levine [10]), ou le taux d'extinction (voir Guo [24], [25], [26]).

Dans [45], une solution stable de (20) s'éteignant en temps fini en un seul point est construite. Son comportement au voisinage du point d'extinction (analogue du temps d'explosion) est décrit avec précision (voir le Théorème de la page 130).

Théorème 4 (Existence d'une solution de (20) s'éteignant en temps fini) *Pour tout $a \in \Omega$, l'équation (20) admet une solution h s'éteignant en temps fini $T > 0$. De plus, $\forall x \in \Omega \backslash \{a\}$, $h(x,t) \to h^*(x)$ quand $t \to T$ et*

$$h^*(x) \sim \left[\frac{(\beta+1)^2|x-a|^2}{8\beta|\log|x-a||}\right]^{\frac{1}{\beta+1}} \quad \textit{quand } x \to a.$$

3 Estimations générales sur les solutions explosives positives de (2)

On se propose maintenant d'affiner les estimations (7) et (8) de Giga et Kohn. Dans ce but, on s'intéresse d'abord au problème de classification des solutions de (6) globales en espace et en temps et uniformément bornées.

Dans [44], le résultat suivant est établi (Théorème 2 page 191):

Théorème 5 (Théorème de Liouville pour (6)) *Soit w une solution de (6) définie pour $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ telle que $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $0 \le w(y,s) \le C$. Alors, on est nécessairement dans l'un des cas suivants:*
i) $w \equiv 0$,
ii) $w \equiv \kappa$ où $\kappa = (p-1)^{-\frac{1}{p-1}}$,
iii) $\exists s_0 \in \mathbb{R}$ tel que $w(y,s) = \varphi(s-s_0)$ où

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}.$$

Remarque: Remarquons que φ est une connexion dans L^∞ des deux points critiques de (6): 0 et κ. En effet,

$$\dot{\varphi} = -\frac{\varphi}{p-1} + \varphi^p,\ \varphi(-\infty) = \kappa,\ \varphi(+\infty) = 0.$$

Remarque: Il suffit d'avoir une solution de (6) définie sur $(-\infty, s^*)$ pour avoir un théorème de classification (voir Corollaire 2 page 191).

À travers la transformation (5), ce Théorème a comme corollaire le résultat suivant (voir Corollaire 3 page 191):

Corollaire 1 *Soit u une solution de (2) définie pour $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ telle que $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $0 \le u(x,t) \le C(T-t)^{-\frac{1}{p-1}}$. Alors,*
soit $u \equiv 0$,
soit $\exists T^* \ge T$ tel que $u(x,t) = \kappa(T^*-t)^{-\frac{1}{p-1}}$.

La preuve du Théorème 5 s'appuie fondamentalement sur les points suivants:

1) une classification des comportements linéaires de $w(s)$ quand $s \to -\infty$ dans $L^2_\rho(\mathbb{R}^N)$ ($L^\infty_{loc}(\mathbb{R}^N)$) où $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}$,

2) l'usage des transformations géométriques

$$w(y,s) \to w_{a,b}(y,s) = w(y + ae^{\frac{s}{2}}, s+b)$$

pour $a \in \mathbb{R}^N$ et $b \in \mathbb{R}$,

3) un critère d'explosion en temps fini au voisinage du point critique κ de la fonctionnelle d'énergie associée à (6):

Si w est solution de (6) définie pour tout temps $s \geq -\log T$ et que pour un certain $s_0 \geq -\log T$, $\int w(y,s_0)\rho(y)dy > \int \kappa\rho(y)dy$, alors $w(s)$ explose en temps fini.
(Proposition 3.5 page 205).

À l'aide d'un argument de compacité, on obtient dans [44] les estimations uniformes suivantes sur les solutions positives de (2) (Théorème 1 page 189)

Théorème 6 (Estimations optimales à l'ordre 0 sur $u(t)$ à l'explosion) *On suppose que Ω est un domaine convexe borné de classe $C^{2,\alpha}$ dans $\mathbb{R}^N$ ou que $\Omega = \mathbb{R}^N$. On considère $u(t)$ une solution de l'équation (2) explosant en temps fini T. Si de plus, $u(0) \geq 0$ et $u(0) \in H^1(\Omega)$, alors,*

$$(T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} \to \kappa$$

$$\text{et } (T-t)^{\frac{1}{p-1}+1}\|\Delta u(t)\|_{L^\infty} + (T-t)^{\frac{1}{p-1}+\frac{1}{2}}\|\nabla u(t)\|_{L^\infty} \to 0 \text{ quand } t \to T.$$

De façon équivalente, pour tout $a \in \Omega$,

$$\|w_a(s)\|_{L^\infty} \to \kappa \text{ et } \|\Delta w_a(s)\|_{L^\infty} + \|\nabla w_a(s)\|_{L^\infty} \to 0 \text{ quand } s \to +\infty.$$

Le Théorème 6 combiné avec des estimations a priori des solutions de (6) dans $W^{3,\infty}(\mathbb{R}^N)$ a permis dans [46] d'affiner les résultats à l'ordre un dans le cas $\Omega = \mathbb{R}^N$ (Théorème 1 page 231):

Théorème 7 (Estimations uniformes optimales à l'ordre un sur les solutions positives de (2)) *Il existe des constantes positives C_1, C_2 et C_3 telles que pour toute solution positive de (2) explosant en temps fini vérifiant $u(0) \in H^1(\mathbb{R}^N)$ et pour tout $\epsilon > 0$, il existe $t_0(\epsilon) < T$ tel que*
i) $\forall t \in [t_0(\epsilon), T)$,

$$\begin{aligned} \|u(t)\|_{L^\infty} &\leq \left(\kappa + (\tfrac{N\kappa}{2p} + \epsilon)\tfrac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}}, \\ \|\nabla^i u(t)\|_{L^\infty} &\leq C_i \frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{i/2}} \end{aligned}$$

pour $i = 1,2,3$,
ii) $\forall s \geq -\log(T - t_0(\epsilon))$, $\forall a \in \mathbb{R}^N$,

$$\|w_a(s)\|_{L^\infty} \leq \kappa + (\frac{N\kappa}{2p} + \epsilon)\frac{1}{s}, \ \|\nabla^i w_a(s)\|_{L^\infty} \leq \frac{C_i}{s^{i/2}}.$$

Remarque: Dans le cas $N = 1$, Herrero et Velázquez [28] (Filippas et Kohn [11] aussi) ont prouvé des estimations reliées au Théorème 7, grâce à une propriété de Sturm utilisée en premier par Chen et Matano [9] (le nombre d'oscillations en espace est une fonction décroissante du temps).
Remarque: Il existe dans [44] et [46] des versions des Théorèmes 6 et 7 valables pour une suite de solutions de (2) et qui donnent de la compacité (Théorème 1' page 190 et Théorème 1' page 232).

Le Théorème 6 nous permet de comparer les tailles relatives des termes de diffusion (Δu) et de réaction (u^p) dans (2) ponctuellement en espace-temps. En effet, on démontre dans [44] le Théorème de localisation suivant (Théorème 3 page 192):

Théorème 8 (Comparaison avec l'équation différentielle ordinaire) *On suppose que Ω est un domaine convexe et borné de classe $C^{2,\alpha}$ ou $\Omega = \mathbb{R}^N$, et que $u_0 \in H^1(\Omega)$. Alors, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ tel que $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \Omega$,*

$$|u_t - u^p| \leq \epsilon u^p + C_\epsilon.$$

Ainsi, la solution de l'équation aux dérivées partielles (2) est comparable uniformément et globalement en espace-temps à une solution de l'équation différentielle ordinaire (localisée par définition)

$$u' = u^p. \tag{21}$$

Ce Théorème constitue ainsi une justification a posteriori du changement de variables (5) qui a permis toute l'étude de (2). En effet, le choix de (5) était en quelque sorte motivé par la recherche d'une comparaison de $u(x,t)$ à $\kappa(T - t)^{-\frac{1}{p-1}}$ qui est justement la solution de (21) qui explose au temps T.
Remarque: De multiples conséquences découlent de ce théorème.
Par exemple (Corollaire 1 de [44], page 188):

Corollaire 2 *On suppose que Ω est un domaine convexe et borné de classe $C^{2,\alpha}$ ou $\Omega = \mathbb{R}^N$. Alors, pour toute solution positive u de (2) qui explose en temps fini T et qui vérifie $u(0) \in H^1(\Omega)$, pour tout $\epsilon_0 > 0$, il existe $t_0(\epsilon_0, u_0) < T$ tel que pour tous $a \in \Omega$ et $t \in [t_0, T)$, si $u(a,t) \leq (1 - \epsilon_0)\kappa(T - t)^{-\frac{1}{p-1}}$, alors a n'est pas point d'explosion de u.*

4 Existence de profil à l'explosion pour les solutions de (2)

Grâce aux estimations de Théorème 7, on démontre dans [46] un Théorème de classification des profils dans la variable $\frac{y}{\sqrt{s}}$, qui sépare l'espace en partie singulière (là où il y a explosion) et partie régulière dans le cas non dégénéré (Théorème 2 page 234).

Théorème 9 (Classification des profils à l'explosion) *Il existe $k \in \{0, 1, ..., N\}$ et une matrice $N \times N$ orthogonale Q tels que $w(Q(z)\sqrt{s}, s) \to f_k(z)$ uniformément sur tout compact $|z| \leq C$, où $f_k(z) = (p - 1 + \frac{(p-1)^2}{4p} \sum_{i=1}^{N-k} |z_i|^2)^{-\frac{1}{p-1}}$ si $k \leq N - 1$ et $f_N(z) = \kappa = (p-1)^{-\frac{1}{p-1}}$.*

Remarque: Ce résultat a été prouvé aussi par Velázquez dans [55]. Cependant, grâce aux techniques uniformes de [46], on peut montrer que la vitesse de convergence est indépendante du point d'explosion considéré, alors qu'elle en dépend dans le résultat de [55].

Un des problèmes intéressants qui en découle est de relier toutes les notions de profils connues: profil pour $|y|$ borné, $|z| = \frac{|y|}{\sqrt{s}}$ borné ou $x \simeq 0$. On démontre dans [46] que ces notions sont équivalentes dans le cas d'une solution qui explose en un point de façon non dégénérée (cas générique), ce qui clarifie de nombreux points évoqués dans des travaux précédents. On a finalement le Théorème suivant (Théorème 3 page 234).

Théorème 10 (Équivalence des comportements explosifs en un point) *Soit $u(t)$ une solution de (2) définie sur $\mathbb{R}^N \times [0,T)$, et $a \in \mathbb{R}^N$. On a l'équivalence des trois comportements suivants de $u(t)$ et de $w_a(s)$ (définie en (5)):*

i) $\forall R > 0, \sup_{|y|\le R} \left| w_a(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ *quand* $s \to +\infty$,

ii) $\forall R > 0, \sup_{|z|\le R} \left| w_a(z\sqrt{s},s) - f_0(z)\right| \to 0$ *quand* $s \to +\infty$ *avec* $f_0(z) = (p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

iii) $\exists \epsilon_0 > 0$ *tel que pour tout* $|x-a| \le \epsilon_0$, $u(x,t) \to u^*(x)$ *quand* $t \to T$ *et* $u^*(x) \sim \left[\frac{8p|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1}{p-1}}$ *quand* $x \to a$.

Dans ce cas, a un point d'explosion isolé de $u(t)$.

Remarque: Grâce aux estimations uniformes utilisées dans la preuve de ce théorème, on peut prouver que les vitesses de convergence dans chaque expression *i*), *ii*) ou *iii*) dépend de la vitesse de convergence dans les deux autres et d'une borne sur $\|u_0\|_{C^2(\mathbb{R}^N)}$ (et non sur u_0). Les estimations de Velázquez [55] permettent aussi d'avoir de résultat d'équivalence, mais les convergences dépendent du point d'explosion considéré.

La thèse est organisée en deux parties:

Première partie: Existence et stabilité de solutions explosives pour des équations de type chaleur et description précise de leur profil à l'explosion.
Organisée en quatre articles [47], [48], [59] et [45] (dont trois en commun avec Frank Merle), elle reprend les résultats de la section 2 de cette introduction.

Deuxième partie: Estimations générales des solutions positives explosives de l'équation de la chaleur non linéaire et notions de profils à l'explosion.
Elle englobe les résultats des sections 3 et 4 de l'introduction, sous la forme de trois articles écrits en collaboration avec Frank Merle ([43], [44] et [46]).

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Première partie

Existence et stabilité de solutions explosives pour des équations de type chaleur et description précise de leur profil à l'explosion

Chapitre 1

Stabilité du profil à l'explosion pour les équations du type $u_t = \Delta u + |u|^{p-1}u$

Équations aux dérivées partielles/*Partial Differential Equations*

Stabilité du profil à l'explosion pour les équations du type $u_t = \Delta u + |u|^{p-1}u$ †

Frank Merle et Hatem Zaag

Résumé - On considère dans cette note l'équation non linéaire suivante:

$$u_t = \Delta u + |u|^{p-1}u, u(.,0) = u_0,$$

(et d'autres extensions de cette équation, où le principe du maximum ne s'applique pas). On décrit d'abord le comportement au voisinage de l'explosion d'une solution explosant en temps fini. Ensuite, on montre que ce comportement est stable.

Stability of blow-up profile for equation of the type $u_t = \Delta u + |u|^{p-1}u$

Abstract - *In this note, we consider the following nonlinear heat equation*

$$u_t = \Delta u + |u|^{p-1}u, u(.,0) = u_0,$$

(and various extensions of this equation, where the maximum principle do not apply). We first describe precisely the behavior of a blow-up solution near blow-up time and blow-up point. We then show a stability result on this behavior.

Abridged English Version - In this note, we consider the following non-linear equation:

$$u_t = \Delta u + |u|^{p-1}u, u(.,0) = u_0 \in H, \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$. We note $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume in addition the exponent p subcritical: if $N \geq 3$ then $1 < p < (N+2)/(N-2)$, otherwise, $1 < p < +\infty$. Other types of equations will be also considered.

We study the case where the solution $u(t)$ of (1) blows-up in finite time in the sense that u exists on $[0,T)$ with $T < +\infty$, and $\|u(t)\|_H \to +\infty$ when $t \to T$. (see Ball [1], Levine [13]). In this case, there is at least one blow-up point a (that is $a \in \mathbb{R}^N$ such that: $|u(a,t)| \to +\infty$ when $t \to T$). We are interested in the structure of the solution near this point. We want to study the behavior of $u(t)$ near blow-up time and point, and the stability of such behavior.

This problem has been extensively studied in the last recent years (see [6], [8], [9], [10]).

In [3], Bricmont and Kupiainen construct a blow-up solution $u(t)$ for (1), which approaches a universal profile with a boundary layer separating regions where $u(x,t)$ is "large" and regions where $u(x,t)$ is "small", and moving at the rate

$$\sqrt{(T-t)|\log(T-t)|}. \tag{2}$$

For that, they used ideas close to the renormalization theory, and some hard analysis.

† Note parue dans C. R. Acad. Sci. Paris Sér. I Math. 322, 1996, pp. 345-350.

In this note, we shall give an idea of a more elementary proof of their result, based on a more geometrical approach and on techniques of a priori estimates, which apply to other contexts. (In particular, we do not use maximum principle).

Theorem 1 Existence of a blow-up solution with a boundary layer with the rate (2)
There exists $T_0 > 0$ *such that for each* $T \in (0, T_0]$, $\forall g \in H$ *with* $\|g\|_{L^\infty} \leq (\log T)^{-2}$, *one can find* $d_0 \in \mathbb{R}$ *and* $d_1 \in \mathbb{R}^N$ *such that for each* $a \in \mathbb{R}^N$, *the solution* $u(t)$ *of equation (1) with initial data*

$$u_0(x) = T^{-\frac{1}{p-1}}\left\{f(z)(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p}|z|^2}) + g(z)\right\},$$

where: $z = (x - a)(|\log T|T)^{-\frac{1}{2}}$, *blows-up at* T *at only one blow-up point:* a. *Moreover,*

$$(3) \qquad \lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f(z)\|_{L^\infty(\mathbb{R}^N)} = 0,$$

with $f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$.

Remark: Such behavior is suspected to be generic.
Remark: Related results using strongly dimension one and maximum principle were obtained in [11] for $N = 1$.

In [19], second author shows analogous results for the following equations (where the maximum principle do not apply):
$u_t = \Delta u + |u|^{p-1}u + i|u|^{r-1}u$, $U_t = \Delta U + |U|^{p-1}U + F_1(U)$ with $U : \mathbb{R}^N \to \mathbb{R}^M$, $|F_1(U)| \leq C|U|^r$, where $p < \frac{N+2}{N-2}$ if $N \geq 3$, and $1 \leq r < p$.

Moreover, we suspect that the same analysis can be carried for other types of equations not satisfying maximum principle, for example: $u_t = -\Delta^2 u + |u|^{p-1}u$.

As in the paper of Bricmont and Kupiainen [2], we won't use maximum principle in the proof. The technique used here will allow us using geometrical interpretation of quantities of the type of d_0 and d_1 to derive stability results concerning this type of behavior, with respect to perturbations of the initial data.

Theorem 2 Stability of the blow-up behavior with respect to initial data
Let $\hat{u}_0$ *be an initial data constructed in Theorem 1. Let* $\hat{u}(t)$ *be the solution of equation (1) with initial data* $\hat{u}_0$, $\hat{T}$ *its blow-up time and* $\hat{a}$ *its blow-up point.*
Then there exists a neighborhood $\mathcal{V}_0$ *of* $\hat{u}_0$ *in* H *which has the following property: For each* u_0 *in* $\mathcal{V}_0$, $u(t)$ *blows-up in finite time* $T = T(u_0)$ *at only one blow-up point* $a = a(u_0)$, *where* $u(t)$ *is the solution of equation (1) with initial data* u_0. *Moreover,* $u(t)$ *behaves near* $T(u_0)$ *and* $a(u_0)$ *in an analogous way as* $\hat{u}(t)$:

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f(z)\|_{L^\infty(\mathbb{R}^N)} = 0.$$

Remark: Theorem 2 yields the fact that the blow-up profile $f(z)$ is stable with respect to perturbations in initial data.
Remark: From [14], we have $T(u_0) \to \hat{T}$, $a(u_0) \to \hat{a}$, as $u_0 \to \hat{u}_0$ in H.

According to a result in [14], we have the following corollary:

Corollary 1 *($N \geq 2$) For arbitrary given set of k points* $x_1, \ldots, x_k$ *in* $\mathbb{R}^N$, *there exists initial data* u_0 *such that the solution* u *of (1) with initial data* u_0 *blows-up exactly at* $x_1, \ldots, x_k$.

I - Introduction - Dans cette note, on considère l'équation de la chaleur non linéaire suivante:

$$u_t = \Delta u + |u|^{p-1}u, u(.,0) = u_0 \in H, \tag{1}$$

avec $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$. On note $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. De plus, on suppose l'exposant p sous-critique: si $N \geq 3$ alors $1 < p < (N+2)/(N-2)$, sinon, $1 < p < +\infty$. D'autres types d'équations seront également considérés.
On étudie le cas où la solution $u(t)$ de (1) explose en temps fini, au sens où u existe sur $[0,T)$ avec $T < +\infty$ et $\|u(t)\|_H \to +\infty$ quand $t \to T$. (voir Ball [1], Levine [13]). Dans ce cas, il existe au moins un point d'explosion a (qui est un point $a \in \mathbb{R}^N$ satisfaisant: $|u(a,t)| \to +\infty$ quand $t \to T$). On s'intéresse à la structure de la solution au voisinage de ce point. On étudie le comportement de $u(t)$ au voisinage du temps et du point d'explosion, ainsi que la stabilité d'un tel comportement.

Ce problème a été beaucoup étudié ces dernières années (voir [6], [8], [9], [10]).

Dans [3], Bricmont et Kupiainen ont construit une solution $u(t)$ de (1), explosant en temps fini, et qui approche un profil universel séparant les régions où $u(x,t)$ est "grande" des régions où $u(x,t)$ est "petite", avec une interface se déplaçant selon

$$\sqrt{(T-t)|\log(T-t)|}. \tag{2}$$

Pour démontrer ce résultat, ils ont utilisé des idées proches de la théorie de la renormalisation, et des estimations assez difficiles.
Dans cette note, on donne une idée d'une preuve plus élémentaire de leur résultat, s'appuyant sur une approche plus géométrique et sur des techniques d'estimations à priori, qui s'appliquent dans d'autres contextes. (En particulier, on n'utilise pas de principe du maximum).

Théorème 1 Existence d'une solution explosant en temps fini avec une interface se déplaçant selon (2)
Il existe $T_0 > 0$ tel que pour tout $T \in (0, T_0]$, $\forall g \in H$ avec $\|g\|_{L^\infty} \leq (\log T)^{-2}$, on peut trouver $d_0 \in \mathbb{R}$ et $d_1 \in \mathbb{R}^N$ tels que pour tout $a \in \mathbb{R}^N$, la solution $u(t)$ de l'équation (1) avec donnée initiale

$$u_0(x) = T^{-\frac{1}{p-1}}\Big\{f(z)(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p}|z|^2}) + g(z)\Big\},$$

avec: $z = (x-a)(|\log T|T)^{-\frac{1}{2}}$, explose en temps fini T en un seul point d'explosion: a. De plus,

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}}u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}}z, t) - f(z)\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{3}$$

avec $f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$.

Remarque: On soupçonne ce comportement d'être générique.
Remarque: Des résultats similaires ont été obtenus dans [11] pour $N = 1$ grâce au principe du maximum et à la dimension un.

Dans [19], le deuxième auteur montre des résultats analogues pour les équations suivantes (où le principe du maximum ne s'applique pas):

$u_t = \Delta u + |u|^{p-1}u + i|u|^{r-1}u$, $U_t = \Delta U + |U|^{p-1}U + F_1(U)$, avec $U : \mathbb{R}^N \to \mathbb{R}^M$, $|F_1(U)| \le C|U|^r$, $p < \frac{N+2}{N-2}$ si $N \ge 3$, et $1 \le r < p$.

De plus, on pense qu'on peut obtenir par les mêmes méthodes des résultats d'explosion pour d'autres équations ne vérifiant pas le principe du maximum, par exemple: $u_t = -\Delta^2 u + |u|^{p-1}u$.

Comme dans [2], on n'utilise pas de principe du maximum dans la preuve. Les techniques utilisées ici permettront grâce à une interpretation géométrique de quantités du type de d_0 et d_1 d'obtenir des résultats de stabilité concernant ce type de comportement par rapport aux données initiales.

Théorème 2 Stabilité du comportement à l'explosion par rapport aux données initiales
Soit $\hat{u}_0$ une donnée initiale construite au Théorème 1. Soit $\hat{u}(t)$ la solution de l'équation (1) avec donnée initiale $\hat{u}_0$, $\hat{T}$ son temps d'explosion, et $\hat{a}$ son point d'explosion.
Alors il existe un voisinage $\mathcal{V}_0$ de $\hat{u}_0$ dans H avec les propriétés suivantes:
Pour tout u_0 dans $\mathcal{V}_0$, $u(t)$ explose en temps fini $T = T(u_0)$ en un seul point d'explosion $a = a(u_0)$, où $u(t)$ est la solution de l'équation (1) avec donnée initiale u_0. De plus, le comportement de $u(t)$ au voisinage de $T(u_0)$ et $a(u_0)$ est analogue au comportement de $\hat{u}(t)$ au voisinage de $\hat{T}$ et $\hat{a}$:

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f(z)\|_{L^\infty(\mathbb{R}^N)} = 0.$$

Remarque: Le Théorème 2 implique que le profil à l'explosion $f(z)$ est stable par rapport à des perturbations dans les données initiales.
Remarque: Après [14], on a $T(u_0) \to \hat{T}$, $a(u_0) \to \hat{a}$, quand $u_0 \to \hat{u}_0$ dans H.

D'après un résultat dans [14], on a le corollaire suivant:

Corollaire 1 *($N \ge 2$) Pour tout ensemble de k points $x_1,\dots, x_k$ dans $\mathbb{R}^N$, il existe une donnée initiale u_0 telle que la solution u de (1) avec donnée initiale u_0 explose exactement en $x_1,\dots, x_k$.*

II - Idées de la démonstration du Théorème 1 - Pour les preuves des autres résultats et pour plus de détails, voir [15].

Partie I: Transformation du problème - Nous traitons le cas $N = 1$. On introduit les *variables auto-similaires*:

$$y = \frac{x-a}{\sqrt{T-t}}, s = -\log(T-t), w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{4}$$

où a est le point d'explosion et T le temps d'explosion de la solution $u(t)$ à construire.
On introduit

$$(5) q(y,s) = w(y,s) - \varphi(y,s) = w(y,s) - \{\frac{\kappa}{2ps} + (p-1+\frac{(p-1)^2}{4p}\frac{y^2}{s})^{-\frac{1}{p-1}}\}.$$

q satisfait l'équation suivante:

$$\frac{\partial q}{\partial s} = (\mathcal{L} + V(y,s))q + E(q,y,s), \tag{6}$$

avec $\mathcal{L}(q) = \frac{\partial^2 q}{\partial y^2} - \frac{1}{2}y\frac{\partial q}{\partial y} + q$, $V(y,s) = p(\varphi^{p-1} - \frac{1}{p-1})$ et
$E(q,y,s) = \{|\varphi+q|^{p-1}(\varphi+q) - \varphi^p - p\varphi^{p-1}q\} + \{\frac{\partial^2\varphi}{\partial y^2} - \frac{1}{2}y\frac{\partial\varphi}{\partial y} - \frac{1}{p-1}\varphi + \varphi^p - \frac{\partial\varphi}{\partial s}\}$.
On écrit alors $q(y,s) = q(y,s)\chi_0(\frac{y}{K_0\sqrt{s}}) + q(y,s)(1-\chi_0(\frac{y}{K_0\sqrt{s}})) = q_b(y,s) + q_e(y,s)$, où $K_0 > 0$, $\chi_0 \in C_0^\infty(\mathbb{R})$, $\chi_0 \equiv 1$ sur $[-1,1]$ et $\chi_0 \equiv 0$ sur $\mathbb{R}\backslash[-2,2]$.

On décompose ensuite q_b suivant le spectre de $\mathcal{L}$ dans $L^2(\mathbb{R}, d\mu)$ avec $d\mu(y) = \frac{e^{y^2/4}}{\sqrt{4\pi}}$ ($spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}$). On obtient:

$$(7) \quad q(y,s) = q_b(y,s) + q_e(y,s) = \{\sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s)\} + q_e(y,s),$$

où h_m est la fonction propre qui correspond à $1 - \frac{m}{2}$, $q_-(y,s)$ est la projection de $q_b(y,s)$ sur l'espace des valeurs propres négatives de $\mathcal{L}$.

On va construire u_0 telle que $u(t)$ vérifie une estimation plus forte que (3). Définissons d'abord pour $A > 0$, $s > 0$:
- $V_A(s) = \{r \in L^2(\mathbb{R}, d\mu) \mid |r_m| \le As^{-2}, m = 0,1; |r_2| \le A^2(\log s)s^{-2}; |r_-(y)| \le A(1+|y|^3)s^{-2}; \|r_e\|_{L^\infty} \le A^2 s^{-\frac{1}{2}}$; avec $r(y) = \sum_{m=0}^{2} r_m h_m(y) + r_-(y) + r_e(y)\}$,
- $\hat{V}_A(s) = [-\frac{A}{s^2}, \frac{A}{s^2}]^2 \subset \mathbb{R}^2$.

On cherche $A > 0$ et $S_0 > 0$ tels que pour tout $s_0 \ge S_0$, $g \in H$ avec $\|g\|_{L^\infty} \le \frac{1}{s_0^2}$, on peut trouver $(d_0, d_1) \in \mathbb{R}^2$ tels que $\forall s \ge s_0$, $\lim_{s\to\infty} \|w_{d_0,d_1}(y,s) - f(\frac{y}{\sqrt{s}})\|_{L^\infty} = \lim_{s\to\infty} \|q_{d_0,d_1}(y,s)\|_{L^\infty} = 0$, où q_{d_0,d_1} est la solution de l'équation (6) avec donnée initiale à $s = s_0$, $q_{d_0,d_1}(y,s_0) =$

$$(8) \qquad (p - 1 + \frac{(p-1)^2}{4ps_0}y^2)^{-\frac{p}{p-1}}(d_0 + d_1 y/\sqrt{s_0}) - \frac{\kappa}{2ps_0} + g(y/\sqrt{s_0}).$$

On va en fait trouver (d_0, d_1) tels que $q_{d_0,d_1}(s) \in V_A(s)$ pour $s \ge s_0$, ce qui entraîne $\lim_{s\to+\infty} \|q_{d_0,d_1}(s)\|_{L^\infty} = 0$. Il est facile de vérifier alors que $u(t)$ explose en temps fini T avec un seul point d'explosion: $x = a$, et vérifie (3).

Partie II: Réduction à un problème de dimension finie - C'est la partie cruciale de la preuve du Théorème 1. Ici, on montre à travers des estimations à priori que pour contrôler $q(s)$ dans $V_A(s)$ ($s \ge s_0$), il suffit de contrôler $(q_0, q_1)(s)$ dans $\hat{V}_A(s)$ (ainsi, on réduit un problème de dimension infinie à un problème de dimension finie).

Proposition 1 (Contrôle de q par (q_0, q_1)) *Il existe $A_1 > 0$ tel que pour tout $A \ge A_1$, il existe $s_1(A) > 0$ tel que pour tout $s_0 \ge s_1(A)$, pour tout $g \in H$ avec $\|g\|_{L^\infty} \le \frac{1}{s_0^2}$, on a la propriété suivante:*

-si (d_0, d_1) est choisi tel que $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$, et,
-si pour $s_* \ge s_0$, on a $\forall s \in [s_0, s_*]$, $q(s) \in V_A(s)$, et $q(s_*) \in \partial V_A(s_*)$, alors
i) $\forall s \in [s_0, s_*]$, $|q_2(s)| \le A^2 s^{-2}\log s - s^{-3}$, $|q_-(y,s)| \le \frac{A}{2}(1+|y|^3)s^{-2}$, $\|q_e(s)\|_{L^\infty} \le \frac{A^2}{2\sqrt{s}}$.
ii) $(q_0(s_*), q_1(s_*)) \in \partial\hat{V}_A(s_*)$ et il existe $\delta_0 > 0$ tel que $\forall \delta \in (0, \delta_0)$, $(q_0, q_1)(s_* + \delta) \notin \hat{V}_A(s_* + \delta)$.

Partie III: Argument topologique en dimension finie - On choisit $A \ge A_1$. On résout maintenant le problème de dimension finie. On remarque par un calcul explicite:

Lemme 1 (Propriété topologique pour $s = s_0$) *Il existe $s_2(A) > 0$ tel que pour tout $s_0 \geq s_2(A)$, pour tout $g \in H$ avec $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, il existe un ensemble $\mathcal{D}_{g,s_0} \subset \mathbb{R}^2$ topologiquement équivalent à un carré, vérifiant la propriété suivante:*

$$q_{d_0,d_1}(s_0) \in V_A(s_0) \text{ si et seulement si } (d_0, d_1) \in \mathcal{D}_{g,s_0}.$$

On fixe $S_0 > \sup(s_1(A), s_2(A))$ et considère $s_0 \geq S_0$. On démontre le Théorème 1 pour A, s_0 et $g \in H$ avec $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$ par un argument topologique.

On procède par l'absurde, et on suppose que pour tout $(d_0, d_1) \in \mathcal{D}_{g,s_0}$, il existe $s > s_0$ tel que $q_{d_0,d_1}(s) \notin V_A(s)$. Soit $s_*(d_0, d_1)$ l'infinimum de tous ces s. Grâce à la Partie II, on peut définir

$$\begin{array}{rcl} \Phi_g : \mathcal{D}_{g,s_0} & \longrightarrow & \partial\mathcal{C} \\ (d_0, d_1) & \longrightarrow & \dfrac{s_*(d_0,d_1)^2}{A}(q_0, q_1)_{d_0,d_1}(s_*(d_0, d_1)) \end{array}$$

où $\mathcal{C}$ est le carré unité de $\mathbb{R}^2$. On démontre alors que Φ_g est continue et que sa restriction à $\partial\mathcal{D}_{g,s_0}$ est homéomorphe à l'identité. Ceci est une contradiction d'après la théorie du degré topologique, donc il existe $(d_0(g), d_1(g))$ tel que $\forall s \geq s_0$, $q_{d_0,d_1}(s) \in V_A(s)$. Ceci termine la preuve.

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Chapitre 2

Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$

Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$ †

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Abstract In this paper, we consider the following nonlinear equation

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0, \end{aligned}$$

(and various extensions of this equation, where the maximum principle do not apply). We first describe precisely the behavior of a blow-up solution near blow-up time and point. We then show a stability result on this behavior.

Mathematics Subject Classification: 35K, 35B35, 35B40
Key words: Blow-up, Profile, Stability

1 Introduction

In this paper, we are concerned with the following nonlinear equation:

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in H, \end{aligned} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, Δ stands for the Laplacian in $\mathbb{R}^N$. We note $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume in addition the exponent p subcritical: if $N \geq 3$ then $1 < p < (N+2)/(N-2)$, otherwise, $1 < p < +\infty$. Other types of equations will be also considered.
Local Cauchy problem for equation (1) can be solved in H . Moreover, one can show that either the solution $u(t)$ exists on $[0, +\infty)$, or on $[0, T)$ with $T < +\infty$. In this former case, u blows-up in finite time in the sense that

$$\|u(t)\|_H \to +\infty \text{ when } t \to T.$$

(Actually, we have both $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ and $\|u(t)\|_{W^{1,p+1}(\mathbb{R}^N)} \to +\infty$ when $t \to T$).
Here, we are interested in blow-up phenomena (for such case, see for example Ball [1], Levine [14]). We now consider a blow-up solution $u(t)$ and note T its blow-up time. One can show that there is at least one blow-up point a (that is $a \in \mathbb{R}^N$ such that: $|u(a,t)| \to +\infty$ when $t \to T$). We will consider in this paper the case of a finite number of blow-up points (see [15]). More precisely, we will focus for simplicity on the case where there is only one blow-up point. We want to study the profile of the solution near blow-up, and the stability of such behavior with respect to initial data.

Standard tools such as center manifold theory have been proven non efficient in this situation (Cf [6] [3]). In order to treat this problem, we introduce

† Article paru dans Duke Math. J. 86, 1997, pp. 143-195.

similarity variables (as in [8]):

$$(2) \qquad y = \frac{x-a}{\sqrt{T-t}},$$
$$s = -\log(T-t),$$
$$(3) \qquad w_{T,a}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t),$$

where a is the blow-up point and T the blow-up time of $u(t)$.
The study of the profile of u as $t \to T$ is then equivalent to the study of the asymptotic behavior of $w_{T,a}$ (or w for simplicity), as $s \to \infty$, and each result for u has an equivalent formulation in terms of w. The equation satisfied by w is the following:

$$(4) \qquad w_s = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w.$$

Giga and Kohn showed first in [8] that for each $C > 0$,

$$\lim_{s \to +\infty} \sup_{|y| \le C} |\, w(y,s) - \kappa| = 0,$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$, which gives if stated for u:

$$\lim_{t \to T} \sup_{|y| \le C} |\, (T-t)^{1/(p-1)} u(a + y\sqrt{T-t}, t) - \kappa \,| = 0.$$

This result was specified by Filippas and Kohn [6] who established that in N dimension, if w doesn't approach κ exponentially fast, then for each $C > 0$

$$\sup_{|y| \le C} |\, w(y,s) - [\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)]| = o(1/s),$$

which gives if stated for u:

$$(5) \quad \sup_{|y| \le C} |\, (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) - [\kappa + \frac{\kappa}{2p|\log(T-t)|}(N - \frac{1}{2}|y|^2)] \,|$$
$$= o((-\log(T-t))^{-1}).$$

Velazquez obtained in [16] a related result, using maximum principle.

Relaying on a numerical study, Berger and Kohn [2] conjectured that in the case of a non exponential decay, the solution u of (1) would approach an explicit universal profile $f(z)$ depending only on p and independent from initial data as follows:

$$(6) \quad (T-t)^{\frac{1}{p-1}} u(a + \sqrt{(T-t)|\log(T-t)|}\, z, t) = f(z) + O((-\log(T-t))^{-1})$$

in L^∞_{loc}, with

$$(7) \qquad f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

This behavior shows that in the case of one isolated blow-up point, there would be a free-boundary moving in (x,t) coordinates at the rate

$$\sqrt{(T-t)|\log(T-t)|}.$$

This free-boundary roughly separates the space into two regions:

1) the singular one, at the interior of the free-boundary, where Δu can be neglected with respect to $|u|^{p-1}u$, so equation (1) behaves like an ordinary differential equation, and blows-up.

2) the regular one, after the free-boundary, where Δu and $|u|^{p-1}u$ are of the same order.

Herrero and Velazquez in [12] and [13] showed in the case of dimension one ($N = 1$) using maximum principle that u behaves in three manners, one of them is the one suggested by Berger and Kohn, and they proved that estimate (6) is true uniformly on z belonging to compact subsets of $\mathbb{R}$ (without estimating the error).

Going further in this direction, Bricmont and Kupiainen construct a solution for (1) satisfying (6) in a global sense. For that, they used on one hand ideas close to the renormalization theory, and on the other hand hard analysis on equation (4).

In this paper, we shall give a more elementary proof of their result, based on a more geometrical approach and on techniques of a priori estimates:

Theorem 1 Existence of a blow-up solution with a free-boundary behavior of the type (6)
There exists $T_0 > 0$ such that for each $T \in (0, T_0]$, $\forall g \in H$ with $\|g\|_{L^\infty} \leq (\log T)^{-2}$, one can find $d_0 \in \mathbb{R}$ and $d_1 \in \mathbb{R}^N$ such that for each $a \in \mathbb{R}^N$, the equation (1) with initial data

$$u_0(x) = T^{-\frac{1}{p-1}}\left\{f(z)(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p}|z|^2}) + g(z)\right\},$$

$$z = (x - a)(|\log T|T)^{-\frac{1}{2}},$$

has a unique classical solution $u(x,t)$ on $\mathbb{R}^N \times [0, T)$ and
i) u has one and only one blow-up point: a
ii) a free-boundary analogous to (6) moves through u such that

$$\lim_{t \to T} (T - t)^{\frac{1}{p-1}} u(a + ((T - t)|\log(T - t)|)^{\frac{1}{2}} z, t) = f(z) \tag{8}$$

uniformly in $z \in \mathbb{R}^N$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

Remark: We took d_0 and d_1 respectively in the direction of $h_0(y) = 1$ and $h_1(y) = y$, the two first eigenfunctions of $\mathcal{L}$ (Cf section 2), but we could have chosen other directions $D_0(y)$ and $D_1(y)$ (see Theorem 2). We can notice that we have a result in $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We can also obtain blow-up results in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If $p < 1 + \frac{4}{N}$, then $f(z) \in H^1$, and we use the same arguments to solve the problem in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If $p \geq 1 + \frac{4}{N}$, the result in H^1 follows directly from the stability result (see Theorem 2 below).
Remark: Such behavior is suspected to be generic.

Remark 1.1

One can ask the following questions:

a) Why does the free-boundary move at such a speed?

b) Why is the profile precisely the function f?

As in various physical situations, we suspect that the asymptotic behavior of $w \to \kappa$ is described by self-similar solutions of equation (4).
Since we are dealing with equation of the heat type (Cf (4)), the natural scaling is $\frac{y}{\sqrt{s}}$. Let us hence try to find a solution of the form $v(\frac{y}{\sqrt{s}})$, with

$$v(0) = \kappa, \lim_{|z| \to \infty} |v(z)| = 0. \tag{9}$$

A direct computation shows that v must satisfy the following equation, for each $s > 0$ and each $z \in \mathbb{R}^N$:

$$-\frac{1}{2s} z.\nabla v(z) = \frac{1}{s}\Delta v(z) - \frac{1}{2} z.\nabla v(z) - \frac{1}{p-1} v(z) + |v(z)|^{p-1} v(z) \tag{10}$$

According to Giga and Kohn [10], the only solutions of (10) are the constant ones: $0, \kappa, -\kappa$, which are ruled out by (9). We can then try to search formally regular solutions of (4) of the form

$$V(y,s) = \sum_{j=0}^{\infty} \frac{1}{s^j} v_j(\frac{y}{\sqrt{s}})$$

and compare elements of order $\frac{1}{s^j}$ (in one dimension, in the positive case for simplicity). We obtain for $j = 0$:

$$0 = -\frac{1}{2} z v_0'(z) - \frac{1}{p-1} v_0(z) + v_0(z)^p,$$

and for $j = 1$ ($z \neq 0$)

$$v_1'(z) + a(z) v_1(z) = b(z)$$

with $a(z) = \frac{2}{z}(\frac{1}{p-1} - p v_0(z)^{p-1})$ and $b(z) = v_0'(z) + \frac{2}{z} v_0''(z)$. The solution for v_0 is given by

$$v_0(z) = (p - 1 + c_0 z^2)^{-\frac{1}{p-1}}$$

for an integration constant $c_0 > 0$. Using this to solve the equation on v_1 yields

$$v_1(z) = v_0(z)^p z^2 [c_1 + \int_1^z \zeta^{-2} v_0(\zeta)^{-p} b(\zeta) d\zeta],$$

for another integration constant c_1. Since we want V to be regular, it is natural to require that v_1 is analytic at $z = 0$. v_1 is regular if and only if the coefficient of ζ in the Taylor expansion of $v_0(\zeta)^{-p} b(\zeta)$ near $\zeta = 0$ is zero which turns to be equivalent to $c_0 = \frac{(p-1)^2}{4p}$ after simple calculation. Therefore, $v_0(z) = (p - 1 + \frac{(p-1)^2}{4p} z^2)^{-\frac{1}{p-1}}$. Hence, the first term in the expansion of V is precisely the profile function f. Carrying on calculus yields:

$$v_1(z) = \frac{p-1}{2p} f(z)^p + \frac{(p-1)^2}{4p} z^2 f(z)^p \log f(z) + c_1 z^2 f(z)^p. \tag{11}$$

We note that $v_1(0) = \frac{\kappa}{2p}$.
Unfortunately, we are not able to calculate every v_j. In conclusion, we take an

other approach to obtain approximate self-similar solutions (see the proof of Theorem 1).

As in the paper of Bricmont and Kupiainen [3], we won't use maximum principle in the proof. The technique used here will allow us using geometrical interpretation of quantities of the type of d_0 and d_1 to derive stability results concerning this type of behavior for the free-boundary, with respect to perturbations of initial data and the equation.

Theorem 2 Stability with respect to initial data of the free boundary behavior
Let $\hat{u}_0$ be initial data constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1) with initial data $\hat{u}_0$, $\hat{T}$ its blow-up time and $\hat{a}$ its blow-up point.
Then there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}_0$ in H which has the following property:
For each u_0 in $\mathcal{V}_0$, $u(t)$ blows-up in finite time $T = T(u_0)$ at only one blow-up point $a = a(u_0)$, where $u(t)$ is the solution of equation (1) with initial data u_0.
Moreover, $u(t)$ behaves near $T(u_0)$ and $a(u_0)$ in an analogous way as $\hat{u}(t)$:

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(a+((T-t)|\log(T-t)|)^{\frac{1}{2}}z,t) = f(z)$$

uniformly in $z \in \mathbb{R}^N$.

Remark: Theorem 2 yields the fact that the blow-up profile $f(z)$ is stable with respect to perturbations in initial data.
Remark: From [15], we have $T(u_0) \to \hat{T}$, $a(u_0) \to \hat{a}$, as $u_0 \to \hat{u}_0$ in H.
Remark: For this theorem, we strongly use a finite dimension reduction of the problem in $\mathbb{R}^{1+N}$, which is the space of liberty degrees of the stability Theorem: (T, a).

Remark 1.2
Theorem 2 is true for a more general $\hat{u}_0$: It is enough that $\hat{u}(t)$ satisfies the key estimate of the proof of Theorem 1.
Remark: Since we do not use the maximum principle, we suspect that such analysis can be carried on for other type of equations, for example:

$$u_t = -\Delta^2 u + |u|^2 u,$$

and

(12) $$u_t = \Delta u + |u|^{p-1}u + i|u|^{r-1}u,$$

where $1 < r < p$ ($p < \frac{N+2}{N-2}$ if $N \geq 3$).
See also for other applications [18].

According to a result of Merle [15], we obtain the following corollary for Theorem 2:

Corollary 1.1 *Let D be a convex set in $\mathbb{R}^N$, or $D = \mathbb{R}^N$. For arbitrary given set of k points $x_1,\ldots,\ x_k$ in D, there exist initial data u_0 such that the solution u of (1) with initial data u_0 (with Dirichlet boundary conditions in the case $D \neq \mathbb{R}^N$) blows-up exactly at $x_1,\ldots,\ x_k$.*

Remark: The local behavior at each blow-up point x_i ($|x - x_i| \leq \rho_i$) is also given by (8).

2 Formulation of the problem

We omit the (T, a) or (d_0, d_1) dependence in what follows to simplify the notation.

2.1 Choice of variables

As indicated before, we use *similarity variables*:

$$y = \frac{x-a}{\sqrt{T-t}},$$

$$s = -\log(T-t),$$

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t).$$

We want to prove for suitable initial data that:

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u\big(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t\big) - f(z)\|_{L^\infty} = 0,$$

or stated in terms of w:

$$\lim_{s\to\infty} \|w(y,s) - f(\frac{y}{\sqrt{s}})\|_{L^\infty} = 0,$$

where

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

We will not study as usually done, this limit difference as $s \to +\infty$

$$w(.,s) - f(\frac{\cdot}{\sqrt{s}}),$$

but we introduce instead:

$$q(y,s) = w(y,s) - [\frac{N\kappa}{2ps} + (p - 1 + \frac{(p-1)^2}{4ps}y^2)^{-\frac{1}{p-1}}]. \tag{13}$$

The added term in (13) can be understood from Remark 1.1. There, we tried to obtain for w an expansion of the form $\sum_{j=0}^{+\infty} \frac{1}{s^j} v_j(\frac{y}{\sqrt{s}})$. We got $v_0 = f$ and for v_1 the expression (11). Hence, it is natural to study the difference $w(y,s) - (v_0(\frac{y}{\sqrt{s}}) + \frac{1}{s} v_1(\frac{y}{\sqrt{s}}))$. Since the expression of v_1 is a bit complicated (see (11)), we study instead $w(y,s) - (v_0(\frac{y}{\sqrt{s}}) + \frac{1}{s} v_1(0))$, which is (13) for $N = 1$.
Now, if we introduce

$$\varphi(y,s) = \frac{N\kappa}{2ps} + f(\frac{y}{\sqrt{s}}) = \frac{N\kappa}{2ps} + (p - 1 + \frac{(p-1)^2}{4ps}|y|^2)^{-\frac{1}{p-1}}, \tag{14}$$

we have

$$q(y,s) = w(y,s) - \varphi(y,s).$$

Thus, the problem in Theorem 1 is to construct a function q satisfying

$$\lim_{s\to+\infty} \|q(.,s)\|_{L^\infty} = 0.$$

From (4) and (13), the equation satisfied by q is the following:
for $s > 0$,

$$\frac{\partial q}{\partial s}(y,s) = \mathcal{L}_V(q)(y,s) + B(q(y,s)) + R(y,s), \tag{15}$$

where

- the linear term is

$$\mathcal{L}_V(q) = \mathcal{L}(q) + V(y,s)q \tag{16}$$

  with

$$\mathcal{L}(q) = \Delta q - \tfrac{1}{2}y.\nabla q + q \text{ and } V(y,s) = p(\varphi^{p-1} - \tfrac{1}{p-1}),$$

- the nonlinear term (quadratic in q for p large) is

$$B(q) = |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \tag{17}$$

- and the rest term involving φ is

$$R(y,s) = \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p - \frac{\partial\varphi}{\partial s}. \tag{18}$$

It will be useful to write equation (15) in its integral form: for each $s_0 > 0$, for each $s_1 \geq s_0$, $q(s_1) =$

$$K(s_1,s_0)q(s_0) + \int_{s_0}^{s_1} d\tau K(s_1,\tau)B(q(\tau)) + \int_{s_0}^{s_1} d\tau K(s_1,\tau)R(\tau), \tag{19}$$

where K is the fundamental solution of the linear operator $\mathcal{L}_V$ defined for each $s_0 > 0$ and for each $s_1 \geq s_0$ by,

$$\partial_{s_1} K(s_1,s_0) = \mathcal{L}_V K(s_1,s_0) \tag{20}$$

$$K(s_0,s_0) = Identity.$$

2.2 Decomposition of q

Since $\mathcal{L}_V$ will play an important role in our analysis, let us point some facts on it.

i) The operator $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{R}^N, d\mu)$ with

$$d\mu(y) = \frac{e^{-\frac{|y|^2}{4}}dy}{(4\pi)^{N/2}}. \tag{21}$$

Note here that there is a weight decaying at infinity. The spectrum of $\mathcal{L}$ is explicit. More precisely,

$$spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

- $N=1$:
All the eigenvalues of $\mathcal{L}$ are simple. For $1-\frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{22}$$

h_m satisfies

$$\int h_n h_m d\mu = 2^n n! \delta_{nm}.$$

(We will note also $k_m = h_m/\|h_m\|^2_{L^2_\mu}$.)

- $N \geq 2$:
We write the spectrum of $\mathcal{L}$ as

$$spec(\mathcal{L}) = \{1 - \frac{m_1 + ... + m_N}{2} | m_1, ..., m_N \in \mathbb{N}\}.$$

For $(m_1, ..., m_N) \in \mathbb{N}$, the eigenfunction corresponding to $1 - \frac{m_1+...+m_N}{2}$ is

$$y \longrightarrow h_{m_1}(y_1)...h_{m_N}(y_N),$$

where h_m is defined in (22). In particular,

*1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is

$$H_0(y) = 1, \tag{23}$$

*$\frac{1}{2}$ is of multiplicity N, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y) | i = 1, ..., N\}$, with $H_{1,i}(y) = h_1(y_i)$; we note

$$H_1(y) = (H_{1,1}(y), ..., H_{1,N}(y)), \tag{24}$$

*0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y) | i, j = 1, ..., N, i \leq j\}$, with $H_{2,ii}(y) = h_2(y_i)$, and for $i < j$, $H_{2,ij}(y) = h_1(y_i)h_1(y_j)$; we note

$$H_2(y) = (H_{2,ij}(y), i \leq j). \tag{25}$$

ii) The potential $V(y,s)$ has two fundamental properties that will influence strongly our analysis.

a) We have $V(.,s) \to 0$ in the $L^2(\mathbb{R}, d\mu)$ when $s \to +\infty$. In particular, the effect of V on the bounded sets or in the "blow-up" region ($|x| \leq C\sqrt{s}$) inside the free boundary will be a "perturbation" of the effect of $\mathcal{L}$.

b) Outside the free boundary, we have the following property:
$\forall \epsilon > 0$, $\exists C_\epsilon > 0$, $\exists s_\epsilon$ such that

$$\sup_{s \geq s_\epsilon, \frac{|y|}{\sqrt{s}} \geq C_\epsilon} |V(y,s) - (-\frac{p}{p-1})| \leq \epsilon$$

with $-\frac{p}{p-1} < -1$.
Since 1 is the biggest eigenvalue of $\mathcal{L}$, we can consider that outside the free boundary, the operator $\mathcal{L}_V$ will behave as one with fully negative spectrum, which simplifies greatly the analysis in this region.

Since the behavior of V inside and outside the free boundary is different, let us decompose q as the following:
Let $\chi_0 \in C_0^\infty([0,+\infty))$, with $\mathrm{supp}(\chi_0) \subset [0,2]$ and $\chi_0 \equiv 1$ on $[0,1]$. We define then

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0 s^{\frac{1}{2}}}\right), \tag{26}$$

where $K_0 > 0$ is chosen large enough so that various technical estimates hold. We write $q = q_b + q_e$ where

$$q_b = q\chi \text{ and } q_e = q(1-\chi).$$

Let us remark that

$$\mathrm{supp}\ q_b(s) \subset B(0, 2K_0\sqrt{s}) \text{ and } \mathrm{supp}\ q_e(s) \subset \mathbb{R}\backslash B(0, K_0\sqrt{s}).$$

Then we study q_b using the structure of $\mathcal{L}$. Since $\mathcal{L}$ has $1+N$ expanding directions (corresponding to eigenvalues 1 and $\frac{1}{2}$) and $\frac{N(N+1)}{2}$ neutral ones, we write q_b with respect to the eigenspaces of $\mathcal{L}$ as follows:

$$q_b(y,s) = \sum_{m=0}^{2} q_m(s).H_m(y) + q_-(y,s) \tag{27}$$

where
$q_0(s)$ is the projection of q_b on H_0,
$q_{1,i}(s)$ is the projection of q_b on $H_{1,i}$, $q_1(s) = (q_{1,i}(s), ..., q_{1,N}(s))$, $H_1(y)$ is given by (24),
$q_{2,ij}(s)$ is the projection of q_b on $H_{2,ij}$, $i \leq j$, $q_2(s) = (q_{2,ij}(s), i \leq j)$, $H_2(y)$ is given by (25),
$q_-(y,s) = P_-(q_b)$ and P_- the projector on the negative subspace of $\mathcal{L}$.

In conclusion, we write q into 5 “components” as follows:

$$q(y,s) = \sum_{m=0}^{2} q_m(s).H_m(y) + q_-(y,s) + q_e(y,s). \tag{28}$$

(Note here that q_m are coordinates of q_b and not of q).
In particular, if $N=1$ and $m = 0,1,2$, $q_m(s)$ and $H_m(y)$ are scalar functions, and $H_m(y) = h_m(y)$. We write in this case:

$$q(y,s) = \sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s) + q_e(y,s). \tag{29}$$

Let us now prove Theorem 1.

3 Existence of a blow-up solution with the given free-boundary profile

This section is devoted to the proof of Theorem 1.

3.1 Transformation of the problem

As in [3], we give the proof in one dimension (same proof holds in higher dimension). We also assume a to be zero, without loss of generality.
Let us consider initial data:

$$u_{0,d_0,d_1}(x) = T^{-\frac{1}{p-1}}\Big\{f(z)(1+\frac{d_0+d_1 z}{p-1+\frac{(p-1)^2}{4p}z^2}) + g(z)\Big\},$$

where

$$z = x(|\log T|T)^{-\frac{1}{2}}.$$

We want to prove first that there exists $T_0 > 0$ such that for each $T \in (0, T_0]$, for every $g \in H$ with $\|g\|_{L^\infty} \le (\log T)^{-2}$, we can find $(d_0, d_1) \in \mathbb{R}^2$ such that

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} u_{d_0,d_1}(((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) = f(z) \tag{30}$$

uniformly in $z \in \mathbb{R}$, where u_{d_0,d_1} is the solution of (1) with initial data u_{0,d_0,d_1}, and

$$f(z) = (p-1+\frac{(p-1)^2}{4p}z^2)^{-\frac{1}{p-1}}. \tag{31}$$

This property will imply that u_{d_0,d_1} blows-up at time T at one single point: $x = 0$. Indeed,

Proposition 3.1 Single blow-up point properties of solutions
Let $u(t)$ be a solution of equation (1). If u satisfies the following property

$$\lim_{t\to T}\|(T-t)^{\frac{1}{p-1}} u(\sqrt{(T-t)|\log(T-t)|}z, t) - f(z)\|_{L^\infty} = 0 \tag{32}$$

then $u(t)$ blows-up at time T at one single point: $x = 0$.

Proof: For each $b \in \mathbb{R}$, we have from (32)

$$\lim_{t\to T}\big\{(T-t)^{\frac{1}{p-1}} u(b,t) - f(\frac{b}{\sqrt{(T-t)|\log(T-t)|}})\big\} = 0.$$

Using (31), we obtain $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(0,t) = \kappa$ and for $b \neq 0$, $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(b,t) = 0$. A result by Giga and Kohn in [8] shows that b is a blow-up point if and only if $\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(b,t) = \pm\kappa$. This concludes the proof of proposition 3.1.

Therefore, it remains to find $(d_0, d_1) \in \mathbb{R}^2$ so that (30) holds to conclude the proof of Theorem 1.
If we use the formulation of the problem in section 2, the problem reduces to find $S_0 > 0$ such that for each $s_0 \ge S_0$, $g \in H$ with $\|g\|_{L^\infty} \le \frac{1}{s_0^2}$, we can find $(d_0, d_1) \in \mathbb{R}^2$ so that the equation (15)

$$\frac{\partial q}{\partial s}(y,s) = \mathcal{L}_V(q)(y,s) + B(q(y,s)) + R(y,s),$$

with initial data at $s = s_0$, $q_{d_0,d_1}(y, s_0) =$

$$(p-1+\frac{(p-1)^2}{4ps_0}y^2)^{-\frac{p}{p-1}}(d_0+d_1y/\sqrt{s_0})-\frac{\kappa}{2ps_0}+g(y/\sqrt{s_0}), \tag{33}$$

has a solution $q(d_0, d_1)$ satisfying

$$\lim_{s\to\infty}\sup_{y\in\mathbb{R}}|q_{d_0,d_1}(y,s)| = 0. \tag{34}$$

q will always depend on g, d_0 and d_1, but we will omit theses dependences in the notations (except when it is necessary).
The convergence of q to zero in $L^\infty(\mathbb{R})$ follows directly if we construct $q(s)$ solution of equation (15) satisfying a geometrical property, that is q belongs to a set $V_A \subset C([s_0, +\infty), L^2(\mathbb{R}, d\mu))$, such that V_A shrinks to $q \equiv 0$ when $s \to \infty$. More precisely we have the following definitions:

Definition 3.1 *For each $A > 0$, for each $s > 0$, we define $V_A(s)$ as being the set of all functions r in $L^2(\mathbb{R}, d\mu)$ such that*

$$\begin{aligned} |r_m(s)| &\le As^{-2}, m = 0, 1, \\ |r_2(s)| &\le A^2(\log s)s^{-2}, \\ |r_-(y,s)| &\le A(1+|y|^3)s^{-2}, \\ \|r_e(s)\|_{L^\infty} &\le A^2 s^{-\frac{1}{2}}, \end{aligned}$$

where $r(y) = \sum_{m=0}^{2} r_m(s)h_m(y) + r_-(y,s) + r_e(y,s)$ (Cf decomposition (29)).

Definition 3.2 *For each $A > 0$, we define V_A as being the set of all functions q in $C([s_0, +\infty), L^2(\mathbb{R}, d\mu))$ satisfying $q(s) \in V_A(s)$ for each $s \ge s_0$.*

Indeed, assume that $\forall s \ge s_0$ $q(s) \in V_A(s)$. Let us show that $\forall s \ge s_0$, $\sup_{y\in\mathbb{R}}|q(y,s)| \le \frac{C(A)}{\sqrt{s}}$, which implies (34).
We have from the definitions of q_b and q_e

$$\begin{aligned} q(y,s) &= q_b(y,s) + q_e(y,s) \\ &= q_b(y,s).1_{\{|y|\le 2K_0\sqrt{s}\}} + q_e(y,s) \\ &= \Big(\sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s)\Big).1_{\{|y|\le 2K_0\sqrt{s}\}}(y,s) + q_e(y,s). \end{aligned}$$

Using the definitions of h_m (Cf (22)) and V_A, the conclusion follows.

3.2 Proof of Theorem 1

Using these geometrical aspects, what we have to do is finally to find $A > 0$ and $S_0 > 0$ such that for each $s_0 \ge S_0$, $g \in H$ with $\|g\|_\infty \le \frac{1}{s_0^2}$, we can find $(d_0, d_1) \in \mathbb{R}^2$ so that $\forall s \ge s_0$,

$$q_{d_0,d_1}(s) \in V_A(s). \tag{35}$$

Let us explain briefly the general ideas of the proof.

-In a first part, we will reduce the problem of controlling all the components of q in V_A to a problem of controlling $(q_0, q_1)(s)$. That is, we reduce an infinite dimensional problem to a finite dimensional one.

-In a second part, we solve the finite dimensional problem, that is to find $(d_0, d_1) \in \mathbb{R}^2$ such that $(q_0, q_1)(s)$ satisfies certain conditions. We will proceed by contradiction and use dynamics in dimension 2 of $(q_0, q_1)(s)$ to reach a topological obstruction (using Index Theory).

The constant C now denotes a universal one independent of variables, only depending upon constants of the problem such as p.

Part I: Reduction to a finite dimensional problem

In this section, we show that finding $(d_0, d_1) \in \mathbb{R}^2$ such that $\forall s \geq s_0$ $q(s) \in V_A(s)$ is equivalent to finding $(d_0, d_1) \in \mathbb{R}^2$ such that $|q_m(s)| \leq \frac{A}{s^2}$ $\forall s \geq s_0$, $\forall m \in \{0, 1\}$. For this purpose, we give the following definition:

Definition 3.3 *For each $A > 0$, for each $s > 0$ we define $\hat{V}_A(s)$ as being the set $[-\frac{A}{s^2}, \frac{A}{s^2}]^2 \subset \mathbb{R}^2$.*
For each $A > 0$, we define $\hat{V}_A$ as being the set of all (q_0, q_1) in $C([s_0, +\infty), \mathbb{R}^2)$ satisfying $(q_0, q_1)(s) \in \hat{V}_A(s)$ $\forall s \geq s_0$.

Step 1: Reduction for initial data

Let us show that for a given A (to be chosen later), for $s_0 \geq s_1(A)$, the control of $q(s_0)$ in $V_A(s_0)$ is equivalent to the control of $(q_0, q_1)(s_0)$ in $\hat{V}_A(s_0)$.

Lemma 3.1 *i) For each $A > 0$, there exists $s_1(A) > 0$ such that for each $s_0 \geq s_1(A)$, $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, then*

$$\begin{aligned} |q_2(s_0)| &\leq (\log s_0) {s_0}^{-2}, \\ |q_-(y, s_0)| &\leq C(1 + |y|^3) {s_0}^{-2}, \\ \|q_e(., s_0)\|_{L^\infty} &\leq {s_0}^{-\frac{1}{2}} \end{aligned}$$

ii) There exists $A_1 > 0$ such that for each $A \geq A_1$, there exists $s_1(A) > 0$ such that for each $s_0 \geq s_1(A)$, $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, we have the following equivalence:

$$q(s_0) \in V_A(s_0) \textit{ if and only if } (q_0, q_1)(s_0) \in \hat{V}_A(s_0).$$

Proof:

We first note that part *ii*) of the lemma follows immediately from part *i*) and definition 3.1. We prove then only part *i*).

Let $A > 0$, $s_0 > 0$ and $g \in H$ such that $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$. Let $(d_0, d_1) \in \mathbb{R}^2$.

We write initial data (Cf (33)) as

$$q(y, s_0) = q^0(y, s_0) + q^1(y, s_0) + q^2(y, s_0) + q^3(y, s_0)$$

where $q^0(y, s_0) = d_0 F(\frac{y}{\sqrt{s_0}})$, $q^1(y, s_0) = d_1 \frac{y}{\sqrt{s_0}} F(\frac{y}{\sqrt{s_0}})$, $q^2(y, s_0) = -\frac{\kappa}{2ps_0}$, $q^3(y, s_0) = g(\frac{y}{\sqrt{s_0}})$ and $F(\frac{y}{\sqrt{s_0}}) = (p - 1 + \frac{(p-1)^2}{4ps_0} y^2)^{-\frac{p}{p-1}}$.

We decompose all the q^i as suggested by (29).

-From $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$ we derive that $|q_0^3(s_0)|+|q_1^3(s_0)|+|q_2^3(s_0)|+\|q_e^3(s_0)\|_{L^\infty} \leq \frac{C}{s_0^2}$, and then, $|q_-^3(y,s_0)| \leq \frac{C}{s_0^2}(1+|y|^3)$.
-Using simple calculations we obtain $|q_0^2(s_0)| \leq \frac{C}{s_0}$,
$q_1^2(s_0) = 0$, $|q_2^2(s_0)| \leq Ce^{-s_0}$, $|q_-^2(y,s_0)| \leq Cs_0^{-2}(1+|y|^3)$ and $\|q_e^2(s_0)\|_{L^\infty} \leq Cs_0^{-1}$.
-For q^0, we have $q_0^0(s_0) = d_0 \int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) \sim d_0C(p)$ $(s_0 \to \infty)$,
$q_1^0(s_0) = 0$, $q_2^0(s_0) = d_0 \int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z^2-2}{8} \sim d_0\frac{C'(p)}{s_0}$ $(s_0 \to \infty)$,
$|q_-^0(y,s_0)| \leq d_0\frac{C}{s_0}(1+|y|^3)$ and $\|q_e^0(s_0)\|_{L^\infty} \leq Cd_0$.
All theses last bounds are simple to obtain, perhaps except that for q_-^0. Indeed, we write $q_-^0(y,s_0) =$
$d_0\chi_{s_0}F(\frac{y}{\sqrt{s_0}}) - d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) - d_0\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z^2-2}{8}(y^2-2)$. The last term can be bounded by $\frac{Cd_0}{s_0}(1+|y|^3)$. We write the first term as
$d_0\Big\{\chi_{s_0}(y)F(\frac{y}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0) - \int d\mu(z)(\chi_{s_0}F(\frac{z}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0))\Big\}$. Using a Lipschitz property, we have $|\chi_{s_0}(y)F(\frac{y}{\sqrt{s_0}}) - \chi_{s_0}(0)F(0)| \leq \frac{Cy^2}{s_0}$, and the conclusion follows.
-Similarly, we obtain for q^1, $q_0^1(s_0) = 0$, $q_1^1(s_0) = \frac{d_1}{\sqrt{s_0}}\int d\mu(z)\chi_{s_0}F(\frac{z}{\sqrt{s_0}})\frac{z}{2}z \sim d_1\frac{C''(p)}{\sqrt{s_0}}$ $(s_0 \to \infty)$, $q_2^1(s_0) = 0$, $|q_-^1(y,s_0)| \leq d_1\frac{C}{s_0^{3/2}}(1+|y|^3)$ and $\|q_e^1(s_0)\|_{L^\infty} \leq C\frac{d_1}{\sqrt{s_0}}$.

Hence, by linearity, we write

$$\begin{aligned} q_0(s_0) &= d_0a_0(s_0) + b_0(g,s_0) \\ q_1(s_0) &= d_1a_1(s_0) + b_1(g,s_0) \end{aligned} \tag{36}$$

with $a_0(s_0) \sim C(p)$, $a_1(s_0) \sim \frac{C''(p)}{\sqrt{s_0}}$, $|b_0(g,s_0)| \leq \frac{C}{s_0}$ and $|b_1(g,s_0)| \leq \frac{C}{s_0^2}$. Therefore, we see that if (d_0,d_1) is chosen such that $(q_0,q_1)(s_0) \in \hat{V}_A(s_0)$ and if $s_0 \geq s_1(A)$, we obtain $|d_m| \leq \frac{C}{s_0}$ for $m \in \{0,1\}$. Using linearity and the above estimates, we obtain $|q_2(s_0)| \leq \frac{C}{s_0^2}$, $|q_-(y,s_0)| \leq \frac{C}{s_0^2}(1+|y|^3)$ and $\|q_e(s_0)\| \leq \frac{C}{s_0}$. Taking $s_1(A)$ larger we conclude the proof of lemma 3.1.

Step 2: A priori estimates
This step is the crucial one in the proof of Theorem 1. Here, we will show through a priori estimates that for $s \geq s_0$, the control of q in $V_A(s)$ reduces to the control of (q_0,q_1) in $\hat{V}_A(s)$. Indeed, this result will imply that if for $s_* \geq s_0$, $q(s_*) \in \partial V_A(s_*)$, then $(q_0(s_*),q_1(s_*)) \in \partial\hat{V}_A(s_*)$. (Compare with definition 3.1).

Remark 3.1
We shall note here that for each initial data $q(s_0)$, equation (15) has a unique solution on $[s_0,S]$ with either $S=+\infty$ or $S<+\infty$ and $\|q(s)\|_{L^\infty} \to +\infty$, when $s \to S$. Therefore, in the case where $S<+\infty$, there exists $s_* > s_0$ such that $q(s_*) \notin V_A(s_*)$ and the solution is in particular defined up to s_*.

Proposition 3.2 (Control of q by (q_0,q_1) in V_A) *There exists $A_2 > 0$ such that for each $A \geq A_2$, there exists $s_2(A) > 0$ such that for each $s_0 \geq s_2(A)$, for each $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, we have the following property:*
-if (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0)) \in \hat{V}_A(s_0)$, and,

-if for $s_1 \geq s_0$*, we have* $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$,
then $\forall s \in [s_0, s_1]$,

$$\begin{aligned} |q_2(s)| &\leq A^2 s^{-2} \log s - s^{-3} \\ |q_-(y,s)| &\leq \frac{A}{2}(1+|y|^3)s^{-2} \\ \|q_e(s)\|_{L^\infty} &\leq \frac{A^2}{2\sqrt{s}}. \end{aligned}$$

Proof: see Proof of Proposition 3.2 below.

Step 3: Transversality

Using now the fact that (q_0, q_1) controls the evolution of q in V_A, we show a transversality condition of (q_0, q_1) on $\partial \hat{V}_A(s_*)$.

Lemma 3.2 *There exists* $A_3 > 0$ *such that for each* $A \geq A_3$*, there exists* $s_3(A)$ *such that for each* $s_0 \geq s_3(A)$*, we have the following properties:*
i) Assume there exists $s_* \geq s_0$ *such that* $q(s_*) \in V_A(s_*)$ *and* $(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*)$*, then there exists* $\delta_0 > 0$ *such that* $\forall \delta \in (0, \delta_0)$, $(q_0, q_1)(s_* + \delta) \notin \hat{V}_A(s_* + \delta)$.
ii) If $q(s_0) \in V_A(s_0)$, $q(s) \in V_A(s)$ $\forall s \in [s_0, s_*]$ *and* $q(s_*) \in \partial V_A(s_*)$ *then there exists* $\delta_0 > 0$ *such that* $\forall \delta \in (0, \delta_0)$, $q(s_* + \delta) \notin V_A(s_* + \delta)$.

Proof:
Part $ii)$ follows from Step 2 and part $i)$.
To prove part $i)$, we will show that for each $m \in \{0,1\}$, for each $\epsilon \in \{-1,1\}$, if $q_m(s_*) = \epsilon \frac{A}{s_*^2}$, then $\frac{dq_m}{ds}(s_*)$ has the opposite sign of $\frac{d}{ds}(\frac{\epsilon A}{s^2})(s_*)$ so that (q_0, q_1) actually leaves $\hat{V}_A$ at s_* for $s_* \geq s_0$ where s_0 will be large. Now, let us compute $\frac{dq_0}{ds}(s_*)$ and $\frac{dq_1}{ds}(s_*)$ for $q(s_*) \in V_A(s_*)$ and $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$. First, we note that in this case, $\|q(s_*)\|_{L^\infty} \leq \frac{CA^2}{\sqrt{s_*}}$ and $|q_b(y, s_*)| \leq CA^2 \frac{\log s_*}{s_*^2}(1+|y|^3)$ (Provided $A \geq 1$). Below, the classical notation $O(l)$ stands for a quantity whose absolute value is bounded precisely by l and not Cl.
For $m \in \{0,1\}$, we derive from equation (15) and (22): $\int d\mu \chi(s_*) \frac{\partial q}{\partial s} k_m =$

$$\int d\mu \chi(s_*) \mathcal{L} q k_m + \int d\mu \chi(s_*) V q k_m + \int d\mu \chi(s_*) B(q) k_m + \int d\mu \chi(s_*) R(s_*) k_m.$$

We now estimate each term of this identity:
a) $|\int d\mu \chi(s_*) \frac{\partial q}{\partial s} k_m - \frac{dq_m}{ds}| = |\int d\mu \frac{d\chi}{ds} q k_m| \leq |\int d\mu \frac{d\chi}{ds} q k_m| \leq \int d\mu |\frac{d\chi}{ds}| \frac{CA^2}{\sqrt{s_*}} |k_m|$
$\leq Ce^{-s_*}$ if $s_0 \geq s_3(A)$.
b) Since $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{R}, d\mu)$, we write

$$\int d\mu \chi(s_*) \mathcal{L} q k_m = \int d\mu \mathcal{L}(\chi(s_*) k_m) q.$$

Using $\mathcal{L}(\chi(s_*) k_m) = (1 - \frac{m}{2}) \chi(s_*) k_m + \frac{\partial^2 \chi}{\partial s^2} k_m + \frac{\partial \chi}{\partial y}(2 \frac{\partial k_m}{\partial y} - \frac{y}{2} k_m)$,
we obtain $\int d\mu \chi(s_*) \mathcal{L} q k_m = (1 - \frac{m}{2}) q_m(s_*) + O(CAe^{-s_*})$.
c) We then have from (16): $\forall y, |V(y,s)| \leq \frac{C}{s}(1+|y|^2)$. Therefore,

$$|\int d\mu \chi(s_*) V q k_m| \leq \int d\mu \frac{C}{s_*}(1+|y|^5) \frac{CA^2 \log s_*}{s_*^2} |k_m| \leq \frac{CA^2 \log s_*}{s_*^3}$$

d) A standard Taylor expansion combined with the definition of V_A shows that $|\chi(y,s_*)B(q(y,s_*))| \leq C|q|^2 \leq C(|q_b|^2+|q_e|^2) \leq \frac{CA^4(\log s_*)^2}{s_*^4}(1+|y|^3)^2 + 1_{\{|y|\geq K\sqrt{s_*}\}}(y)\frac{A^2}{\sqrt{s_*}}$. Thus, $|\int d\mu\chi(s_*)B(q)k_m| \leq \frac{CA^4(\log s_*)^2}{s_*^4} + Ce^{-s_*}$.
e) A direct calculus yields $|\int d\mu\chi(s_*)R(s_*)k_m| \leq \frac{C(p)}{s_*^2}$ (Actually it is equal to 0 if $m=1$). Indeed, in the case $m=0$, we start from (18) and (14) and expand each term up to the second order when $s\to\infty$. Since $\varphi(y,s) = f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps}$, we derive:
1) $\int d\mu\chi(s)(-\frac{\varphi}{p-1}) = -\frac{1}{p-1}(\kappa - \frac{\kappa}{2ps} + \frac{\kappa}{2ps} + O(Cs^{-2})) = -\frac{\kappa}{p-1} + O(Cs^{-2})$,
2) $\int d\mu\chi(s)\varphi^p = \int d\mu f^p + \frac{\kappa}{2ps}\int d\mu p f^{p-1} + O(Cs^{-2}) = \frac{\kappa}{p-1} - \frac{\kappa}{2(p-1)s} + \frac{\kappa}{2ps}\frac{p}{p-1} + O(Cs^{-2}) = \frac{\kappa}{p-1} + O(Cs^{-2})$,
3) $\varphi_s(y,s) = \frac{p-1}{4ps^2}y^2 f^p - \frac{\kappa}{2ps^2}$ and then $\int d\mu\chi(s)(-\varphi_s) = O(Cs^{-2})$,
4) $\varphi_y(y,s) = -\frac{p-1}{2ps}yf^p$ and then $\int d\mu\chi(s)(-\frac{1}{2}y\varphi_y) = \frac{\kappa}{2ps} + O(Cs^{-2})$,
5) $\varphi_{yy}(y,s) = -\frac{p-1}{2ps}f^p + \frac{(p-1)^2}{4ps^2}y^2 f^{2p-1}$, then $\int d\mu\chi(s)\varphi_{yy} = -\frac{\kappa}{2ps} + O(Cs^{-2})$.
Adding all these expansions, we obtain $\int d\mu\chi_{s_*}R(s_*) = O(C(p)s_*^{-2})$. Concluding steps a) to e), we obtain

$$\frac{dq_m}{ds}(s_*) = (1-\frac{m}{2})\frac{\epsilon A}{s_*^2} + O(\frac{C(p)}{s_*^2}) + O(CA^4\frac{\log s_*}{s_*^3})$$

whenever $q_m(s_*) = \frac{\epsilon A}{s_*^2}$. Let us now fix $A \geq 2C(p)$, and then we take $s_3(A)$ larger so that for $s_0 \geq s_3(A)$, $\forall s \geq s_0$, $\frac{C(p)}{s^2} + O(CA^4\frac{\log s}{s^3}) \leq \frac{3C(p)}{2s^2}$. Hence, if $\epsilon=-1$, $\frac{dq_m}{ds}(s_*) < 0$, if $\epsilon = 1$, $\frac{dq_m}{ds}(s_*) > 0$. This concludes the proof of lemma 3.2.

Now, let us fix $A \geq \sup(A_2, A_3)$.

Part II: Topological argument
Now, we reduce the problem to studying a two-dimensional one. Let us study now this problem. We give its initialization in the following lemma:

Lemma 3.3 (Initialization of the finite dimensional problem) *There exists $s_4(A) > 0$ such that for each $s_0 \geq s_4(A)$, for each $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, there exists a set $\mathcal{D}_{g,s_0} \subset \mathbb{R}^2$ topologically equivalent to a square with the following property:*

$$q(d_0,d_1,s_0) \in V_A(s_0) \textit{ if and only if } (d_0,d_1) \in \mathcal{D}_{g,s_0}.$$

Proof:
As stated by lemma 3.1 *(ii)*, if we take $s_0 > s_1(A)$ and $g \in H$ with $\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$, then it is enough to prove that there exists a set $\mathcal{D}_{g,s_0}$ topologically equivalent to a square satisfying

$$(q_0,q_1)(s_0) \in \hat{V}_A(s_0) \text{ if and only if } (d_0,d_1) \in \mathcal{D}_{g,s_0}.$$

If we refer to the calculus of $q_m(s_0)$ (Cf (36) and what follows), and take $s_4(A) \geq s_0(A)$ and $s_4(A)$ large enough, then this concludes the proof of lemma 3.3.

Now, we fix $S_0 > \sup(s_1(A), s_2(A), s_3(A), s_4(A))$ and take $s_0 \geq S_0$. Then we start the proof of Theorem 1 for A and $s_0(A)$ and a given $g \in H$ with

$\|g\|_{L^\infty} \leq \frac{1}{s_0^2}$.
We argue by contradiction: According to lemma 3.3, for each $(d_0, d_1) \in \mathcal{D}_{g,s_0}$ $q(d_0, d_1, s_0) \in V_A(s_0)$. We suppose then that for each $(d_0, d_1) \in \mathcal{D}_{g,s_0}$, there exists $s > s_0$ such that $q(d_0, d_1, s) \notin V_A(s)$. Let $s_*(d_0, d_1)$ be the infimum of all these s. (Note here that $s_*(d_0, d_1)$ exists because of remark 3.1).

Applying proposition 3.2, we see that $q(d_0, d_1, s_*(d_0, d_1))$ can leave $V_A(s_*(d_0, d_1))$ only by its first two components, hence,

$$(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \in \partial\hat{V}_A(s_*(d_0, d_1)).$$

Therefore, we can define the following function:

$$\begin{array}{rcl} \Phi_g : \mathcal{D}_{g,s_0} & \longrightarrow & \partial\mathcal{C} \\ (d_0, d_1) & \longrightarrow & \dfrac{s_*(d_0, d_1)^2}{A}(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \end{array}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Now, we claim

Proposition 3.3 *i)* Φ_g *is a continuous mapping from* $\mathcal{D}_{g,s_0}$ *to* $\partial\mathcal{C}$.
ii) The restriction of Φ_g *to* $\partial\mathcal{D}_{g,s_0}$ *is homeomorphic to identity.*

From that, a contradiction follows (Index Theory). This means that there exists $(d_0(g), d_1(g))$ such that $\forall s \geq s_0$, $q(d_0, d_1, s) \in V_A(s)$, that is $q \in V_A$. In particular,

$$\|q(s)\|_{L^\infty} \leq \frac{C(A)}{\sqrt{s}}.$$

Using Proposition 3.1, this concludes the proof of Theorem 1.

Proof of Proposition 3.3:
Step 1: i)
We have $(q_0, q_1)(s)$ is a continuous function of $(w(s_0), s) \in H \times [s_0, +\infty)$ where $w(s_0)$ is initial data for equation (4). Since $w(s_0)$ ($= q(y, s_0) + \varphi(y, s_0)$, Cf (33) and (14)) is continuous in (d_0, d_1) (it is linear), we have $(q_0, q_1)(s)$ is continuous with respect to (d_0, d_1, s). Now, using the transversality property of (q_0, q_1) on $\partial\hat{V}_A$ (lemma 3.2), we claim that $s_*(d_0, d_1)$ is continuous. Therefore, Φ_g is continuous.
Step 2: ii)
If $(d_0, d_1) \in \partial\mathcal{D}_{g,s_0}$, then, according to the proof of lemma 3.3, $(q_0, q_1)(s_0) \in \partial\hat{V}_A(s_0)$. Therefore, using $q(s_0) \in V_A(s_0)$ (lemma 3.1), we have $q(s_0) \in \partial V_A(s_0)$. Applying *ii)* of lemma 3.2 with s_0 and $s_* = s_0$ yields $\delta_0 > 0$ such that $\forall\delta \in (0, \delta_0)$, $q(s_0 + \delta) \notin V_A(s_0 + \delta)$. Hence,

$$s_*(d_0, d_1) = s_0,$$

and $\Phi_g(d_0, d_1) = \frac{s_0^2}{A}(q_0, q_1)(s_0)$. Formulas (36) show then that $\Phi_{g|\partial\mathcal{D}_{g,s_0}}$ is homeomorphic to identity. This concludes the proof of Proposition 3.3. Let us now prove Proposition 3.2.

3.3 Proof of Proposition 3.2

For further purpose, we are going to prove a more general proposition which implies Proposition 3.2.

Proposition 3.4 *For each $\tilde{A} > 0$ There exists $\tilde{A}_2(\tilde{A}) > 0$ such that for each $A \geq \tilde{A}_2(\tilde{A})$, there exists $\tilde{s}_2(\tilde{A}, A) > 0$ such that for each $s_0 \geq \tilde{s}_2(\tilde{A}, A)$, for each solution q of equation (15), we have the following property:*
-if

$$(37)\qquad \begin{array}{rcl} |q_m(s_0)| & \leq & A s_0^{-2}, m = 0, 1 \\ |q_2(s_0)| & \leq & \tilde{A} s_0^{-2} \log s_0, \\ |q_-(y, s_0)| & \leq & \tilde{A} s_0^{-2}(1 + |y|^3), \\ \|q_e(s)\|_{L^\infty} & \leq & \tilde{A} s_0^{-1/2}, \end{array}$$

-if for $s_1 \geq s_0$, we have $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$, then $\forall s \in [s_0, s_1]$,

$$\begin{array}{rcl} |q_2(s)| & \leq & A^2 s^{-2} \log s - s^{-3} \\ |q_-(y, s)| & \leq & \frac{A}{2}(1 + |y|^3) s^{-2} \\ \|q_e(s)\|_{L^\infty} & \leq & \frac{A^2}{2\sqrt{s}}. \end{array}$$

Proposition 3.4 implies Proposition 3.2. Indeed, referring to Lemma 3.1, we apply proposition 3.4 with $\tilde{A} = \max(1, C)$. This gives $\tilde{A}_2 > 0$, and for each $A \geq \tilde{A}_2$, $\tilde{s}_2(\tilde{A}, A)$. If we take $s_2(A) = \max(\tilde{s}_2(\max(1, C), A), s_1(A))$ (Cf Lemma 3.1), then, applying proposition 3.4 and Lemma 3.1, one easily checks that Proposition 3.2 is valid for these values.

Proof of Proposition 3.4
The proof is divided in two parts:

In a first part, we give a priori estimates on $q(s)$ in $V_A(s)$: assume that for given $A > 0$ large, $\tilde{A} > 0$, $\rho > 0$ and initial time $s_0 \geq s_5(A, \tilde{A}, \rho)$, we have $q(s) \in V_A(s)$ for each $s \in [\sigma, \sigma + \rho]$, where $\sigma \geq s_0$. Using the equation satisfied by q, we then derive new bounds on q_2, q_- and q_e in $[\sigma, \sigma + \rho]$ (involving A, $\tilde{A}$ and ρ).

In a second part, we will use these new bounds to conclude the proof of Proposition 3.4.

Step 1: A priori estimates of q.
Let us recall the integral equation satisfied by q (Cf (19)):

$$(38)\qquad q(s) = K(s, \sigma) q(\sigma) + \int_\sigma^s d\tau K(s, \tau) B(q(\tau)) + \int_\sigma^s d\tau K(s, \tau) R(\tau),$$

where

$$\begin{array}{rcl} B(q) & = & |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1} q, \\ R(y, s) & = & \Delta\varphi - \frac{1}{2} y.\nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p - \frac{\partial\varphi}{\partial s}, \end{array}$$

and K is the fundamental solution of $\mathcal{L}_V$ (Cf (16)).
We now assume that for each $s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$. Using (38), we derive new bounds on the three terms in the right hand side of (38), and then on q.
In the case $\sigma = s_0$, from initial data properties, it turns out that we obtain better estimates for $s \in [s_0, s_0+\rho]$.
More precisely, we have the following lemma:

Lemma 3.4 *There exists $A_5 > 0$ such that for each $A \geq A_5$, $\tilde{A} > 0$, $\rho^* > 0$, there exists $s_5(A, \tilde{A}, \rho^*) > 0$ with the following property:*
$\forall s_0 \geq s_5(A, \tilde{A}, \rho^*)$, $\forall \rho \leq \rho^*$, assume $\forall s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$ with $\sigma \geq s_0$.
I)Case $\sigma \geq s_0$:
we have $\forall s \in [\sigma, \sigma+\rho]$,
i) **(linear term)**

$$
\begin{aligned}
|\alpha_2(s)| &\leq A^2 \frac{\log \sigma}{s^2} + (s-\sigma) C A s^{-3}, \\
|\alpha_-(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)} A + e^{-(s-\sigma)^2} A^2)(1+|y|^3) s^{-2}, \\
\|\alpha_e(s)\|_{L^\infty} &\leq C(A^2 e^{-\frac{(s-\sigma)}{p}} + A e^{(s-\sigma)}) s^{-\frac{1}{2}},
\end{aligned}
$$

where

$$
K(s,\sigma) q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2} \alpha_m(s) h_m(y) + \alpha_-(y,s) + \alpha_e(y,s).
$$

ii) **(nonlinear term)**

$$
\begin{aligned}
|\beta_2(s)| &\leq \frac{(s-\sigma)}{s^{3+1/2}}, \\
|\beta_-(y,s)| &\leq (s-\sigma)(1+|y|^3) s^{-2-\epsilon}, \\
\|\beta_e(s)\|_{L^\infty} &\leq (s-\sigma) s^{-\frac{1}{2}-\epsilon},
\end{aligned}
$$

where

$$
\epsilon = \epsilon(p) > 0,
$$

and

$$
\int_\sigma^s d\tau K(s,\tau) B(q(\tau)) = \beta(y,s) = \sum_{m=0}^{2} \beta_m(s) h_m(y) + \beta_-(y,s) + \beta_e(y,s).
$$

iii) **(corrective term)**

$$
\begin{aligned}
|\gamma_2(s)| &\leq (s-\sigma) C s^{-3}, \\
|\gamma_-(y,s)| &\leq (s-\sigma) C (1+|y|^3) s^{-2}, \\
\|\gamma_e(s)\|_{L^\infty} &\leq (s-\sigma) s^{-3/4},
\end{aligned}
$$

where

$$
\int_\sigma^s d\tau K(s,\tau) R(.,\tau) = \gamma(y,s) = \sum_{m=0}^{2} \gamma_m(s) h_m(y) + \gamma_-(y,s) + \gamma_e(y,s).
$$

II) Case $\sigma = s_0$:
Assume in addition that $q(s_0)$ satisfies (37). Then, $\forall s \in [s_0, s_0+\rho]$,
i) **(linear term)**

$$\begin{aligned} |\alpha_2(s)| &\leq \tilde{A}\frac{\log s_0}{s^2} + C\max(A,\tilde{A})(s-s_0)s^{-3}, \\ |\alpha_-(y,s)| &\leq C\tilde{A}(1+|y|^3)s^{-2}, \\ \|\alpha_e(s)\|_{L^\infty} &\leq C\tilde{A}(1+e^{(s-s_0)})s^{-\frac{1}{2}}. \end{aligned}$$

We will give the proof of this lemma later.

Step 2: Lemma 3.4 implies Proposition 3.4
Let $\tilde{A}$ be an arbitrary positive number. Let $A > \tilde{A}_2(\tilde{A})$ where $\tilde{A}_2(\tilde{A})$ will be defined later. Let $s_0 > 0$ to be chosen larger than $\tilde{s}_2(A)$ (where $\tilde{s}_2(A)$ will be defined later). Let q be a solution of equation (15) satisfying (37), and $s_1 \geq s_0$. Assume in addition that $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$.
We want to prove that $\forall s \in [s_0, s_1]$

$$(39) \quad |q_2(s)| \leq A^2\frac{\log s}{s^2} - \frac{1}{s^3}, |q_-(y,s)| \leq \frac{A}{2s^2}(1+|y|^3), \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}}.$$

Let $\rho_1 \geq \rho_2$ two positive numbers (to be fixed in terms of A later). It is then enough to prove (39), on one hand for $s - s_0 \leq \rho_1$, and on the other hand for $s - s_0 \geq \rho_2$. In both cases, we use lemma 3.4. Hence, we suppose $A \geq A_5$, $s_0 \geq \max(s_5(A,\tilde{A},\rho_1), s_5(A,\tilde{A},\rho_2))$.

Case 1: $s - s_0 \leq \rho_1$.
Since we have $\forall \tau \in [s_0, s]$, $q(\tau) \in V_A(\tau)$, we apply lemma 3.4 (IIi), Iii), iii)) with A, $\rho^* = \rho_1$ and $\rho = s - s_0$. From (38), we obtain:

$$(40) \quad \begin{aligned} |q_2(s)| &\leq \tilde{A}\frac{\log s_0}{s^2} + C_1(\max(A,\tilde{A})+1)(s-s_0)s^{-3} + (s-s_0)s^{-3-1/2} \\ |q_-(y,s)| &\leq (C_1\tilde{A} + C_1(s-s_0))(1+|y|^3)s^{-2} + (s-s_0)(1+|y|^3)s^{-2-\epsilon} \\ \|q_e(s)\|_{L^\infty} &\leq (C_1\tilde{A} + C_1\tilde{A}e^{s-s_0})s^{-\frac{1}{2}} + (s-s_0)s^{-3/4} + (s-s_0)s^{-\frac{1}{2}-\epsilon}. \end{aligned}$$

To have (39), it is enough to satisfy

$$(41) \quad \begin{aligned} \tilde{A}\frac{\log s_0}{s^2} &\leq \frac{A^2}{2}\frac{\log s}{s^2} \\ C_1\tilde{A}s^{-2} + C_1(s-s_0)s^{-2} &\leq \frac{A}{4}s^{-2} \\ C_1\tilde{A}s^{-1/2} + C_1\tilde{A}e^{s-s_0}s^{-1/2} &\leq \frac{A^2}{4}s^{-\frac{1}{2}}, \end{aligned}$$

on one hand, and

$$(42) \quad \begin{aligned} C_1(\max(A,\tilde{A})+1)(s-s_0)s^{-3} + (s-s_0)s^{-3-1/2} &\leq \frac{A^2}{2}\frac{\log s}{s^2} - s^{-3} \\ (s-s_0)s^{-2-\epsilon} &\leq \frac{A}{4}s^{-2} \\ (s-s_0)s^{-3/4} + (s-s_0)s^{-\frac{1}{2}-\epsilon} &\leq \frac{A^2}{4}s^{-\frac{1}{2}} \end{aligned}$$

on the other hand.
If we restrict ρ_1 to satisfy $C_1\rho_1 \leq \frac{A}{8}$, $C_1\tilde{A}e^{\rho_1} \leq \frac{A^2}{8}$, (which is possible if we fix $\rho_1 = \frac{3}{2}\log A$ for A large), and A to satisfy $\tilde{A} \leq A$, $\tilde{A} \leq \frac{A^2}{2}$, $C_1\tilde{A} \leq \frac{A}{8}$ and $C_1\tilde{A} \leq \frac{A^2}{8}$ (that is $A \geq A_6(\tilde{A})$), then, since $s - s_0 \leq \rho_1$, (41) is satisfied.
With this value of ρ_1, (42) will be satisfied if the following is true:

$$\begin{aligned} C_1(A+1)\frac{3}{2}\log A s^{-3} + \frac{3}{2}\log A s^{-3-1/2} &\leq \frac{A^2}{2}\frac{\log s}{s^2} - s^{-3} \\ \frac{3}{2}\log A s^{-2-\epsilon} &\leq \frac{A}{4}s^{-2} \\ \frac{3}{2}\log A s^{-3/4} + \frac{3}{2}\log A s^{-\frac{1}{2}-\epsilon} &\leq \frac{A^2}{4}s^{-\frac{1}{2}}, \end{aligned}$$

which is possible, if $s_0 \geq s_6(A)$.
This concludes Case 1.

Case 2: $s - s_0 \geq \rho_2$.
Since we have $\forall \tau \in [\sigma, s]$, $q(\tau) \in V_A(\tau)$, we apply Part I) of lemma 3.4 with A, $\rho = \rho^* = \rho_2$, $\sigma = s - \rho_2$. From (38), we derive:

$$\begin{aligned} (43)\quad |q_2(s)| &\leq A^2\frac{\log(s-\rho_2)}{s^2} + C_2A\rho_2 s^{-3} + C_2\rho_2 s^{-3} + \rho_2 s^{-3-1/2} \\ |q_-(y,s)| &\leq C_2(e^{-\frac{1}{2}\rho_2}A + e^{-\rho_2{}^2}A^2 + \rho_2)(1+|y|^3)s^{-2} + \rho_2(1+|y|^3)s^{-2-\epsilon} \\ \|q_e(s)\|_{L^\infty} &\leq C_2(A^2e^{-\frac{\rho_2}{p}} + Ae^{\rho_2})s^{-\frac{1}{2}} + \rho_2 s^{-3/4} + \rho_2 s^{-\frac{1}{2}-\epsilon}, \end{aligned}$$

To obtain (39), it is enough to have:

$$(44)\qquad \begin{aligned} f_{A,\rho_2}(s) &\geq 0 \\ C_2(e^{-\frac{1}{2}\rho_2}A + e^{-\rho_2{}^2}A^2 + \rho_2) &\leq \frac{A}{4} \\ C_2(A^2e^{-\frac{\rho_2}{p}} + Ae^{\rho_2}) &\leq \frac{A^2}{4}, \end{aligned}$$

with

$$f_{A,\rho_2}(s) = A^2\frac{\log s}{s^2} - s^{-3} - [A^2\frac{\log(s-\rho_2)}{s^2} + C_2(A+1)\rho_2 s^{-3} + \rho_2 s^{-3-1/2}]$$

on one hand, and

$$(45)\qquad \begin{aligned} \rho_2 s^{-2-\epsilon} &\leq \frac{A}{4}s^{-2} \\ \rho_2 s^{-3/4} + \rho_2 s^{-\frac{1}{2}-\epsilon} &\leq \frac{A^2}{4}s^{-\frac{1}{2}}, \end{aligned}$$

on the other hand.
Now, it is convenient to fix the value of ρ_2 such that $C_2Ae^{\rho_2} = \frac{A^2}{8}$, that is $\rho_2 = \log\frac{A}{8C_2}$. The conclusion follows from this choice, for A large. Indeed, for arbitrary A, we write

$$|f_{A,\log\frac{A}{8C_2}}(s) - s^{-3}(A^2\log\frac{A}{8C_2} - 1 - C_2(A+1)\log\frac{A}{8C_2})| \leq \frac{CA^2}{s^{3+1/2}}(\log\frac{A}{8C_2})^2.$$

Then, we take $A \geq A_7$ such that

$$\begin{array}{rcl} (A^2 \log \dfrac{A}{8C_2} - 1 - C_2(A+1)\log \dfrac{A}{8C_2}) & \geq & 1 \\ C_2((\dfrac{A}{8C_2})^{-1/2}A + e^{-(\log \frac{A}{8C_2})^2}A^2 + \log \dfrac{A}{8C_2}) & \leq & \dfrac{A}{4} \\ C_2(A^2(\dfrac{A}{8C_2})^{-1/p} + A\dfrac{A}{8C_2}) & \leq & \dfrac{A^2}{4}. \end{array}$$

After, we introduce $s_7(A) > 0$ such that for $s \geq s_0 \geq s_7(A)$, we have $s^{-3-1/2}CA^2(\log \frac{A}{8C_2})^2 \leq \frac{1}{2}s^{-3}$ and (45) satisfied.
This way, (44) and (45) are satisfied, for $A \geq A_7$ and $s_0 \geq s_7(A)$, which concludes Case 2.

We remark that for $A \geq A_8$, we have $\rho_1 = \frac{3}{2}\log A \geq \rho_2 = \log \frac{A}{8C_2}$.
If now we take $A_2 = \sup(A_5, A_6(\tilde{A}), A_7, A_8)$, and then $s_2 = \max(s_5(A,\tilde{A},\rho_1(A)), s_5(A,\tilde{A},\rho_2(A)), s_6(A), s_7(A))$, then this concludes the proof of Proposition 3.2.

Proof of Lemma 3.4
Let $A \geq A_5$ with $A_5 > 0$ to be fixed later. Let $\tilde{A} > 0$, $\rho^* > 0$. We take $\rho \leq \rho^*$ and $s_0 \geq s_5(A,\tilde{A},\rho^*)$. We consider $\sigma \geq s_0$ such that $\forall s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$. For each part $Ii), ii), iii)$ and $IIi)$, we want to find $s_5(A,\tilde{A},\rho_0)$ such that the concerned part holds for $s_0 \geq s_5(A,\tilde{A},\rho^*)$.
The proof is given in two steps:

-In a first step, we give various estimates on different terms appearing in the equation (19).

-In a second step, we use these estimates to conclude the proof.

Step 1: Estimates for equation (38)

i) Estimates on K:

Lemma 3.5 (Bricmont-Kupiainen) .
a) $\forall s \geq \tau \geq 1$ *with* $s \leq 2\tau$, $\forall y, x \in \mathbb{R}$,
$|K(s,\tau,y,x)| \leq Ce^{(s-\tau)\mathcal{L}}(y,x)$, *with*
$e^{\theta\mathcal{L}}(y,x) = \frac{e^{\theta}}{\sqrt{4\pi(1-e^{-\theta})}}\exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}]$.

b) For each $A' > 0$, $A'' > 0$, $A''' > 0$, $\rho^* > 0$, *there exists* $s_9(A', A'', A''', \rho^*)$ *with the following property:*
$\forall s_0 \geq s_9$, *assume that for* $\sigma \geq s_0$,

(46)
$$\begin{array}{rcl} |q_m(\sigma)| & \leq & A'\sigma^{-2}, m = 0,1, \\ |q_2(\sigma)| & \leq & A''(\log \sigma)\sigma^{-2}, \\ |q_-(y,\sigma)| & \leq & A'''(1+|y|^3)\sigma^{-2}, \\ \|q_e(\sigma)\|_{L^\infty} & \leq & A''\sigma^{-\frac{1}{2}}, \end{array}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$|\alpha_2(s)| \quad \leq \quad A''\frac{\log \sigma}{s^2} + (s-\sigma)C\max(A', A''')s^{-3},$$

$$\begin{aligned}|\alpha_-(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A''' + e^{-(s-\sigma)^2}A'')(1+|y|^3)s^{-2},\\ \|\alpha_e(s)\|_{L^\infty} &\leq C(A''e^{-\frac{(s-\sigma)}{p}} + A'''e^{(s-\sigma)})s^{-\frac{1}{2}},\end{aligned}$$

where

$$K(s,\sigma)q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2}\alpha_m(s)h_m(y) + \alpha_-(y,s) + \alpha_e(y,s). \tag{47}$$

c) $\forall\rho^* > 0$, $\exists s_{10}(\rho^*)$ *such that* $\forall\sigma \geq s_{10}(\rho^*)$, $\forall s \in [\sigma, \sigma+\rho^*]$,

$$\begin{aligned}|\gamma_2(s)| &\leq (s-\sigma)Cs^{-3},\\ |\gamma_-(y,s)| &\leq (s-\sigma)C(1+|y|^3)s^{-2},\end{aligned}$$

where

$$\int_\sigma^s d\tau K(s,\tau)R(\tau) = \gamma(y,s) = \sum_{m=0}^{2}\gamma_m(s)h_m(y) + \gamma_-(y,s) + \gamma_e(y,s).$$

Proof:
see Appendix A.
Using the above lemma and simple calculation, we derive the following:

Corollary 3.1 $\forall s \geq \tau \geq 1$ *with* $s \leq 2\tau$, $|\int K(s,\tau,y,x)(1+|x|^m)dx| \leq$

$$C\int e^{(s-\tau)\mathcal{L}}(y,x)(1+|x|^m)dx \leq e^{s-\tau}(1+|y|^m). \tag{48}$$

ii) Estimates on B:

Lemma 3.6 $\forall A > 0$, $\exists s_{11}(A)$ *such that* $\forall\tau \geq s_{11}(A)$, $q(\tau) \in V_A(\tau)$ *implies*

$$|\chi(y,\tau)B(q(y,\tau))| \leq C|q|^2 \tag{49}$$

and

$$|B(q)| \leq C|q|^{\bar{p}} \tag{50}$$

with $\bar{p} = \min(p,2)$.

Proof: Let $A > 0$. If $q(\tau) \in V_A(\tau)$, then $\|q(\tau)\|_{L^\infty} \leq C(A)\tau^{-1/2} \leq \frac{1}{2}f(2K_0)$, if $\tau \geq s_{11}(A)$ (Cf Definition 3.2, (7) for f and (26) for K_0).
(49) and (50) are equivalent to 1), 2) and 3), with

1) $p \geq 2$ and $|B(q)| \leq C|q|^2$,
2) $p < 2$ and $|\chi(y,\tau)B(q(y,\tau))| \leq C|q|^2$,
3) $p < 2$ and $|B(q)| \leq C|q|^p$.

We prove 1), 2) and 3).
For 1), we Taylor expand $B(q)$, and use the boundedness of $|\varphi|$ and $|q|$.
2) holds if $\chi(y,\tau) = 0$. Otherwise, we have $|y| \leq 2K_0\sqrt{\tau}$. Again, we Taylor expand $B(q)$: $\chi(y,\tau)|B(q)| \leq C\chi(y,\tau)|q|^2\int_0^1(1-\theta)|\varphi+\theta q|^{p-2}d\theta$, and conclude writing $\chi(y,s)|\varphi+\theta q|^{p-2} \leq \chi(y,s)(|\varphi|-|q|)^{p-2} \leq (f(2K_0)-\frac{1}{2}f(2K_0))^{p-2} = C$.
For 3), we write $\frac{B(q)}{|q|^p} = \frac{|1+\xi|^{p-1}(1+\xi)-1-p\xi}{|\xi|^p}$ by setting $\xi = \frac{q}{\varphi}$. We easily check that this expression is bounded for $\xi \to 0$ and $\xi \to \infty$.

iii) Estimate on R:

Lemma 3.7 $\exists s_{12} > 0\ \forall \tau \geq s_{12}$,

$$|R(y,\tau)| \leq \frac{C}{\tau}. \tag{51}$$

Proof:
From (18) and (14), we compute: $\varphi_{yy} = -\frac{p-1}{2p\tau} f^p + \frac{(p-1)^2}{4p\tau^2} y^2 f^{2p-1}$, $\varphi_s = -\frac{p-1}{4p\tau^2} y^2 f^p + \frac{\kappa}{2p\tau^2}$, and $\varphi^p - \frac{\varphi}{p-1} - \frac{1}{2} y\varphi_y = [f + \frac{\kappa}{2p\tau}]^p - \frac{\kappa}{2p(p-1)\tau} - \frac{f}{p-1} + \frac{p-1}{4p\tau} y^2 f^p = -\frac{\kappa}{2p(p-1)\tau} + [f + \frac{\kappa}{2p\tau}]^p - f^p$, using a Lipschitz property and simple calculations, the conclusion follows.

iv) Estimates on q in V_A:
From Definition 3.2, we simply derive the following:

Lemma 3.8 $\exists s_{13} > 0\ \forall A > 0,\ \forall \tau \geq s_{13}$, *if* $q(\tau) \in V_A(\tau)$, *then*

$$|q(y,\tau)| \leq CA^2 \tau^{-2} \log \tau (1 + |y|^3) \tag{52}$$

and

$$|q(y,\tau)| \leq CA^2 \tau^{-1/2}. \tag{53}$$

Step 2: Conclusion of the Proof of Lemma 3.4

We choose $s_0 \geq \rho^*$ in all cases so that if $s_0 \leq \sigma \leq \tau \leq \sigma + \rho$ and $\rho \leq \rho^*$, we have $\sigma^{-1} \leq 2s^{-1}$ and $\tau^{-1} \leq 2s^{-1}$.

Ii) linear term in I):
We apply b) of lemma 3.5 with $A' = A$, $A'' = A^2$ and $A''' = A$. Take $s_5(A, \rho^*) = s_9(A, A^2, A, \rho^*)$.

IIi) linear term in II):
We apply b) of lemma 3.5 with $A' = A$, $A'' = \tilde{A}$ and $A''' = \tilde{A}$.

Iii) nonlinear term:
-$\beta_2(s)$:
By definition, $\beta_2(s) = \int d\mu(y) k_2(y) \chi(y,s) \beta(y,s)$.
$= \int d\mu(y) k_2(y) \chi(y,s) \int_\sigma^s d\tau \int K(s,\tau,y,x) B(q(x,\tau)) dx = I + II$, where
$I = \int d\mu(y) k_2(y) \chi(y,s) \int_\sigma^s d\tau \int K(s,\tau,y,x) \chi(x,\tau) B(q(x,\tau)) dx$, and
$II = \int d\mu(y) k_2(y) \chi(y,s) \int_\sigma^s d\tau \int K(s,\tau,y,x)(1 - \chi(x,\tau)) B(q(x,\tau)) dx$.
For I we write:
$|I| \leq \int d\mu(y) |k_2(y)| \int_\sigma^s d\tau \int |K(s,\tau,y,x)| \chi(x,\tau) |B(q(x,\tau))| dx$
$\leq C \int d\mu(y) |k_2(y)| \int_\sigma^s d\tau \int |K(s,\tau,y,x)| |q(x,\tau)|^2 dx$ (Cf (49))
$\leq C \int d\mu(y) |k_2(y)| \int_\sigma^s d\tau \int |K(s,\tau,y,x)| A^4 \tau^{-4} (\log \tau)^2 (1 + |x|^6) dx$ (Cf (52))
$\leq CA^4 \int d\mu(y) |k_2(y)| \int_\sigma^s d\tau \tau^{-4} (\log \tau)^2 e^{s-\tau} (1 + |y|^6)$ (Cf corollary 3.1)
$\leq CA^4 \int d\mu(y) |k_2(y)| (1 + |y|^6)(s - \sigma) \sigma^{-4} (\log s)^2 e^{s-\sigma}$
$\leq CA^4 (s - \sigma) e^{s-\sigma} (\frac{s}{2})^{-4} (\log s)^2$ (we take $s_0 \geq \rho^*$ so that $s \leq \sigma + \rho^* \leq \sigma + s_0 \leq \sigma + \sigma = 2\sigma$)
For II, we use (50) and (53) to have:
$|II| \leq C \int e^{-\frac{y^2}{4}} dy \chi(y,s) |k_2(y)| \int_\sigma^s d\tau \int dx (1 - \chi(x,\tau))$

$\frac{e^{s-\sigma}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\exp[-\frac{(ye^{-(s-\tau)/2}-x)^2}{4(1-e^{-(s-\tau)})}]A^{2\bar{p}}\tau^{-\bar{p}/2}$.

Now, we have $e^{\frac{1}{2}[-\frac{y^2}{4}-\frac{(ye^{-t/2}-x)^2}{4(1-e^{-t})}]} \leq e^{-c(K_0)s} \leq e^{-Cs}$, for $|y| \leq 2K_0\sqrt{s}$ and $|x| \geq K_0\sqrt{\tau}$ (if $s_0 \geq \rho^*$). Hence, we derive

$|II| \leq C\int e^{-\frac{y^2}{8}}dy|k_2(y)|\int_\sigma^s d\tau\int dx(1-\chi(x,\tau))$

$\frac{e^{s-\sigma}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}\exp[-\frac{1}{2}\frac{(ye^{-(s-\tau)/2}-x)^2}{4(1-e^{-(s-\tau)})}]e^{-Cs}A^{2\bar{p}}\tau^{-\bar{p}/2}$.

Using a variable change in x, and carrying all calculation, we bound $|II|$ by $(s-\sigma)e^{-Cs}$, for $s \geq s_{14}(A,\rho^*)$. Adding the bounds for I and II, and taking $\sigma \geq s_{15}(A,\rho^*)$, we obtain the estimate for $\beta_2(s)$.

-$\beta_-(y,s)$:

Using (50), (52), and (48), and computing as before yields $|\beta(y,s)| \leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)^{\bar{p}}(\frac{\log s}{s^2})^{\bar{p}}$. If we multiply this term by $\chi(s)$ and bound in it $|y|^{3\bar{p}-3}$ by $(\sqrt{s})^{3\bar{p}-3}$, we obtain

$|\beta_b(y,s)| \leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)(\sqrt{s})^{3\bar{p}-3}(\frac{\log s}{s^2})^{\bar{p}}$, hence

$|\beta_b(y,s)| \leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}(1+|y|^3)\frac{(\log s)^{\bar{p}}}{s^{(\bar{p}+3)/2}}$, which implies simply the estimate for β_- (for $\sigma \geq s_{16}(\rho^*)$ and some $\epsilon_1(p)$).

-$\beta_e(y,s)$:

Using (50), (53), and (48), and computing as before yields $|\beta(y,s)| \leq CA^{2\bar{p}}(s-\sigma)e^{(s-\sigma)}s^{-\frac{1}{2}\bar{p}}$. From this, we derive directly the estimate for β_e (for $\sigma \geq s_{17}(\rho^*)$ and some $\epsilon_2(p)$).

Finally, we take $\sigma \geq \max(s_{15},s_{16},s_{17})) = s_5(A,\rho^*)$ and $\epsilon = \min(\epsilon_1,\epsilon_2)$ to have the conclusion.

iii) corrective term:

For γ_2 and γ_-, we use c) of lemma 3.5. For γ_e, we start from (51) and write $\gamma_e(y,s) = (1-\chi(y,s))\gamma(y,s) = (1-\chi)\int_\sigma^s d\tau\int dxK(s,\tau,y,x)R(x,\tau)$, and then as in *ii*), $|\gamma_e(y,s)| \leq C\int_\sigma^s d\tau\int dx e^{(s-\tau)\mathcal{L}}(y,x)\frac{C}{\tau} = C\int_\sigma^s \frac{d\tau}{t}e^{s-\tau} \leq \frac{C}{s}(s-\sigma)e^{s-\sigma} \leq (s-\sigma)s^{-\frac{3}{4}}$, if $\sigma \geq s_{10}(\rho^*)$.

4 Stability

In this section, we give the proof of Theorem 2. As in section 3, we consider $N=1$ for simplicity, but the same proof holds in higher dimension. We will mention at the end of the section how to adapt the proof to the case $N \geq 2$.

4.1 Case $N = 1$:

Let us consider $\hat{u}_0$ an initial data in H, constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1):

$$u_t = \Delta u + |u|^{p-1}u, u(0) = \hat{u}_0.$$

Let $\hat{T}$ be its blow-up time and $\hat{a}$ be its blow-up point.

We know from (35) that there exists $\hat{A} > 0$, $\hat{s}_0 > \log\hat{T}$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$, where $\hat{q}_{\hat{T},\hat{a}}$ is defined in (13) by:

$$\hat{q}_{\hat{T},\hat{a}}(y,s) = e^{-\frac{s}{p-1}}\hat{u}(\hat{a}+ye^{-\frac{s}{2}},\hat{T}-e^{-s}) - [\frac{\kappa}{2ps}+(p-1+\frac{(p-1)^2}{4ps}y^2)^{-\frac{1}{p-1}}].$$

Remark: Following Remark 1.2, we can consider a more general $\hat{u}_0$, that is $\hat{u}_0$ with the following property:
$\exists(\hat{T}, \hat{a})$, $\exists \hat{A}, \hat{s}_0$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$. From Definition 3.2, the definition of $\hat{q}_{\hat{T},\hat{a}}(s)$, and Proposition 3.1, $\hat{u}(t)$ blows up at time $\hat{T}$ at one single point $\hat{a}$, and behaves as the conclusion of Theorem 1.

We want to prove that there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}_0$ in H with the following property:

$\forall u_0 \in \mathcal{V}_0$, $u(t)$ blows-up in finite time T at only one blow-up point a, where $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$. Moreover, $u(t)$ satisfies:

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z \ , t) = f(z) \tag{54}$$

uniformly in $z \in \mathbb{R}$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p} z^2)^{-\frac{1}{p-1}}.$$

The proof relays strongly on the same ideas as the proof of Theorem 1: use of finite dimensional parameters, reduction to a finite dimensional problem and continuity. For Theorem 2, we introduce a one-parameter group, defined by:

$$(T, a) \longrightarrow q_{T,a},$$

where $q_{T,a}$ is defined by (13), for a given solution $u(t)$ of equation (1) with initial data u_0. This one-parameter group has an important property: $\forall (T, a)$, $q_{T,a}$ is a solution of equation (15). Therefore, our purpose is to fine-tune the parameter (T, a) in order to get $(T(u_0), a(u_0))$ such that $q_{T(u_0),a(u_0)}(s) \in V_{A_0}(s)$, for $s \geq s_0$, A_0 and s_0 are to be fixed later. Hence, through the reduction to a finite dimensional problem, we give a geometrical interpretation of our problem, since we deal with finite dimensional functions depending on finite dimensional parameters through a one-parameter group.

As indicated in the formulation of the problem in section 2 and used in section 3 (Definitions 3.1 and 3.2), it is enough to prove the following:

Proposition 4.1 (Reduction) *There exist $A_0 > 0$, $s_0 > 0$, D_0 neighborhood of $(\hat{T}, \hat{a})$ in $\mathbb{R}^2$, and $\mathcal{V}_0$ neighborhood of $\hat{u}_0$ in H with the following property: $\forall u_0 \in \mathcal{V}_0, \exists (T, a) \in D_0$ such that $\forall s \geq s_0$, $q_{T,a}(s) \in V_{A_0}(s)$, where $q_{T,a}$ is defined by (13), and $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$. (We keep here the $(T,\ a)$ dependence for clearness).*

Indeed, once this proposition is proved, (54) follows directly from (3), (13) and definitions 3.1, 3.2. Proposition 3.1 applied to $u(x - a, t)$ then shows directly that $u(t)$ blows-up at time T at one single point: $x = a$.

The proof relays strongly on the same ideas as those developed in section 3, and geometrical interpretation of T and a. Let us explain briefly its main ideas:

-In a first part, as before, we reduce the control of all the components of q to a problem of control $(q_0, q_1)(s)$, uniformly for $u_0 \in \mathcal{V}_1$ and $(T, a) \in D_1$ (where $\mathcal{V}_1$ and D_1 are respectively neighborhoods of $\hat{u}_0$ and $(\hat{T}, \hat{a})$).

-In a second part, we focus on the finite dimensional variable $(q_0, q_1)(s)$, and try to control it. We study the behavior of $\hat{q}_{T,a}$ under perturbations in (T, a) near $(\hat{T}, \hat{a})$ (and some topological structure related to these). We then extend the properties of $\hat{q}$ to q, for u_0 near $\hat{u}_0$. We conclude the proof proceeding by contradiction to reach a topological obstruction (using Index Theory).

The constant C again denotes a universal one independent of variables, only depending upon constants of the problem such as p.

For each initial data u_0, $u(t)$ denotes the solution of (1) satisfying $u(0) = u_0$, and for each $(T, a) \in \mathbb{R}^2$, $w_{T,a}$ and $q_{T,a}$ denote the auxiliary functions derived from u by transformations (3) and (13).

Part I: Initialization and reduction to a finite dimensional problem
In this section, we first use continuity arguments to show that for A, s_0 large enough (to be fixed later), for (u_0, T, a) close to $(\hat{u}_0, \hat{T}, \hat{a})$, $q_{T,a}$ is defined at $s = s_0$, and satisfies $q_{T,a}(s_0) \in V_A(s_0)$ (Step 1). After, we aim at finding (T, a) such that $q_{T,a}(s)$ in $V_A(s)$ for $s \geq s_0$. For this purpose, we reduce through a priori estimates the control of $q_{T,a}(s)$ in $V_A(s)$ to the control of $(q_{0,T,a}, q_{1,T,a})(s)$ in $\hat{V}_A(s)$ for $s \geq s_0$ (Step 2).

Step 1: Initialization
We use here the fact that $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$ for any $s \geq \hat{s}_0$, and the continuity of $q_{T,a}$ with respect to initial data u_0 and (T, a), to insure that for fixed $s_0 \geq \hat{s}_0$, $q_{T,a}(s_0) \in V_{2\hat{A}}(s_0)$, for (u_0, T, a) close to $(\hat{u}_0, \hat{T}, \hat{a})$. Hence, if A is large enough, we have $q_{T,a}(s_0) \in V_A(s_0)$ and $q_{T,a}(s_0)$ is "small" in a way.

Lemma 4.1 (Initialization) *For each $s_0 > \hat{s}_0$ there exist $\mathcal{V}_1$ neighborhood of $\hat{u}_0$ in H and $D_1(s_0)$ neighborhood of $(\hat{T}, \hat{a})$ in $\mathbb{R}^2$, such that for each $u_0 \in \mathcal{V}_1$, $(T, a) \in D_1(s_0)$, $q(T, a, s)$ is defined (at least) for $s \in (-\log T, s_0]$, and $q_{T,a}(s_0) \in V_{2\hat{A}}(s_0)$.*

Proof of Lemma 4.1:
$\forall T > 0$, $\forall a \in \mathbb{R}$, $q_{T,a}(s)$ is defined on:
$(-\log T, +\infty)$, if $T \leq \hat{T}$, or $(-\log T, -\log(T - \hat{T}))$, if $T > \hat{T}$.
Therefore, $q_{T,a}(s)$ is defined on $(-\log T, s_0]$ for T near $\hat{T}$.

i) Reduction to the continuity of $q_{T,a}(s_0) \in L^\infty(\mathbb{R})$
Let $s_0 > \hat{s}_0$. It is enough to prove that $\forall \epsilon > 0$, there exist $\mathcal{V}$ and D such that $\forall u_0 \in \mathcal{V}$, $(T, a) \in D$,

$$\|q_{T,a}(s_0) - \hat{q}_{\hat{T},\hat{a}}(s_0)\|_{L^\infty(\mathbb{R})} \leq \epsilon. \tag{55}$$

Indeed, if it is the case, then,

$$\forall m \in \{0,1,2\}, |q_{m,T,a}(s_0) - \hat{q}_{m,\hat{T},\hat{a}}(s_0)| \leq C\epsilon, \tag{56}$$
$$|q_{-,T,a}(y, s_0) - \hat{q}_{-,\hat{T},\hat{a}}(y, s_0)| \leq C\epsilon(1 + |y|^2), \tag{57}$$
$$\|q_{e,T,a}(s_0) - \hat{q}_{e,\hat{T},\hat{a}}(s_0)\|_{L^\infty(\mathbb{R})} \leq C\epsilon. \tag{58}$$

(56) and (58) follow directly from (55). For (57), write
$q_-(y, s) = \chi(y, s)q(y, s) - \sum_{m=0}^{2} q_m(s)h_m(y)$, and use (55) and (56).
Using $\hat{q}_{\hat{T},\hat{a}}(s_0) \in V_{\hat{A}}(s_0)$ and taking $\epsilon > 0$ small enough yields the conclusion of lemma 4.1.

ii) Continuity of $q_{T,a}(s_0) \in L^\infty(\mathbb{R})$
We have:

$$\begin{array}{rcl} q_{T,a}(y,s_0) - \hat{q}_{\hat{T},\hat{a}}(y,s_0) & = & w_{T,a}(y,s_0) - \hat{w}_{\hat{T},\hat{a}}(y,s_0) \\ = e^{-\frac{s_0}{p-1}}\{u(e^{-s_0/2}y+a, T-e^{-s_0}) & - & \hat{u}(e^{-s_0/2}y+\hat{a}, \hat{T}-e^{-s_0})\} \\ = e^{-\frac{s_0}{p-1}}\{u(e^{-s_0/2}y+a, T-e^{-s_0}) & - & \hat{u}(e^{-s_0/2}y+a, T-e^{-s_0})\} \\ +e^{-\frac{s_0}{p-1}}\{\hat{u}(e^{-s_0/2}y+a, T-e^{-s_0}) & - & \hat{u}(e^{-s_0/2}y+\hat{a}, T-e^{-s_0})\} \\ +e^{-\frac{s_0}{p-1}}\{\hat{u}(e^{-s_0/2}y+\hat{a}, T-e^{-s_0}) & - & \hat{u}(e^{-s_0/2}y+\hat{a}, \hat{T}-e^{-s_0})\}. \end{array}$$

Since $u_0 \to u(t) \in \mathcal{C}^1([\frac{\hat{T}-e^{-\hat{s}_0}}{2}, \hat{T} - \frac{e^{-s_0}}{2}], \mathcal{C}^1(\mathbb{R}))$ is defined and continuous (for u_0 near $\hat{u}_0$), we have the conclusion.

Step 2: Uniform finite dimensional reduction
This step is similar to Step 2 of Part 1 in the proof of Theorem 1. Here we show that for A and s_0 to be fixed later, if $q_{T,a}(s_0)$ is "small" in $V_A(s_0)$, then, the control of $q_{T,a}(s)$ in $V_A(s)$ for $s \geq s_0$ reduces to the control of $(q_{0,T,a}, q_{1,T,a})(s)$ in $\hat{V}_A(s)$.

Lemma 4.2 (Control of q by (q_0, q_1) in V_A) *There exists $A_2 > 2\hat{A}$ such that for each $A \geq A_2$, there exists $s_2(A) > 0$ such that for each $s_0 \geq s_2(A)$, we have the following properties:*
i) For any q, solution of equation (15), satisfying
- $q(s_0) \in V_{2\hat{A}}(s_0)$ and,
- for $s_1 \geq s_0$, $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$,
we have: $\forall s \in [s_0, s_1]$,

$$\begin{array}{rcl} |q_2(s)| & \leq & A^2 s^{-2}\log s - s^{-3} \\ |q_-(y,s)| & \leq & \dfrac{A}{2}(1+|y|^3)s^{-2} \\ \|q_e(s)\|_{L^\infty} & \leq & \dfrac{A^2}{2\sqrt{s}}. \end{array}$$

Moreover,
ii) For any q, solution of equation (15), satisfying
- $q(s_0) \in V_{2\hat{A}}(s_0) (\subset V_A(s_0))$,
- For $s_* > s_0$, $q(s) \in V_A(s)$ $\forall s \in [s_0, s_*]$, and
-$q(s_*) \in \partial V_A(s_*)$,
we have $(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*)$, and there exists $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0)$, $(q_0, q_1)(s_* + \delta) \notin \hat{V}_A(s_* + \delta)$, (hence, $q(s_* + \delta) \notin V_A(s_* + \delta)$).

Proof:
i) We apply Proposition 3.4 with $\tilde{A} = \max(2\hat{A}, (2\hat{A})^2)$, and take $A_2 = \max(\tilde{A}_2, 2\hat{A})$, and $s_2(A) = \max(\hat{s}_0 + 1, \tilde{s}_2(\tilde{A}, A))$ to have the conclusion.
ii) We apply *i)* with $s_1 = s_*$, and use Definition 3.1. Then, we apply lemma 3.2.

Part II: Topological argument
Below, we use the notations $q_{T,a}(s) = q(T,a,s)$, $q_{T,a}(y,s) = q(T,a,y,s)$, $q_{m,T,a}(s) = q_m(T,a,s)$.

In Part 1, we have reduced the problem to a finite dimensional one: for each u_0 close to $\hat{u}_0$, we have to find a parameter $(T,a) = (T(u_0), a(u_0))$ near $(\hat{T},\hat{a})$ such that $(q_0,q_1)(T,a,s) \in V_A(s)$ for $s \geq s_0$. We first study the behavior of $\hat{q}(T,a)$ for (T,a) close to $(\hat{T},\hat{a})$. Then, we show a stability result on this behavior for u_0 near $\hat{u}_0$. Therefore, for a given u_0, we proceed by contradiction to prove Proposition 4.1, which implies Theorem 2.

Step 1: Study of $\hat{q}(T,a)$

We study the behavior of $\hat{q}(T,a)$ for (T,a) close to $(\hat{T},\hat{a})$ in $\mathbb{R}^2$.

Proposition 4.2 (Behavior of $\hat{q}(T,a)$ near $(\hat{T},\hat{a})$) *There exists $A_4 > 0$ such that for each $A \geq A_4$, there exists $s_4(A) > 0$ with the following property:*
For each $s_0 \geq s_4(A)$, there exists $D_4(s_0)$ neighborhood of $(\hat{T},\hat{a})$ such that for each $(T,a) \in D_4(s_0) \backslash \{(\hat{T},\hat{a})\}$,
i) $\hat{q}(T,a,s)$ is defined for $s \in (-\log T, s_0]$ and $\hat{q}(T,a,s_0) \in V_A(s_0)$,
ii) $\exists s_*(T,a) > s_0$ such that $\forall s \in [s_0, s_*(T,a)]$, $\hat{q}(T,a,s) \in V_A(s)$ and $\hat{q}(T,a,s_*(T,a)) \in \partial V_A(s_*(T,A))$, and if we define

$$(59) \qquad \begin{array}{rcl} \Psi_{\hat{u}_0} : D_4(s_0)\backslash\{(\hat{T},\hat{a})\} & \longrightarrow & \mathbb{R}^2 \\ (T,a) & \longrightarrow & \dfrac{\hat{s}_*(T,a)^2}{A}(\hat{q}_0,\hat{q}_1)(T,a,\hat{s}_*(T,a)) \end{array}$$

then $Im(\Psi_{\hat{u}_0}) \subset \partial\mathcal{C}$, where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Moreover,
iii)$\Psi_{\hat{u}_0}$ is continuous,
iv) $\forall\epsilon > 0$, there exists a curve $\Gamma_\epsilon \in D_4(s_0)$ such that $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$, and $\forall (T,a) \in \Gamma_\epsilon$, $|(T,a) - (\hat{T},\hat{a})| \leq \epsilon$.

Proof:
In order to prove $i), ii)$, and $iii)$, we take $A \geq A_5$ with $A_5 = \max(2\hat{A}, A_2, A_3)$, $s_0 \geq s_5(A) = \max(\hat{s}_0 + 1, s_2(A), s_3(A))$, $D_5(s_0) = D_1(s_0)$ (with the notations of lemma 4.1). For such A and s_0, we can apply lemma 4.1, and lemma 4.2.

Proof of i):
By lemma 4.1, $\forall (T,a) \in D_5(s_0)$, $\hat{q}(T,a,s)$ is defined (at least) for $s \in (-\log T, s_0]$ and $\hat{q}(T,a,s_0) \in V_{2\hat{A}}(s_0) \subset V_A(s_0)$, which proves $i)$.

Proof of ii):
We claim that $\forall (T,a) \in D_5(s_0)\backslash\{(\hat{T},\hat{a})\}$, $\exists s(T,a) > s_0$such that $\hat{q}(T,a,s) \notin V_A(s)$. Indeed:

Case 1: $T > \hat{T}$:
Since $\hat{q}(T,a,y,s) = e^{-\frac{s}{p-1}}\hat{u}(a + ye^{-\frac{s}{2}}, T - e^{-s}) - \varphi(y,s)$, $\hat{q}(T,a,s)$ is defined on $[s_0, -\log(T-\hat{T}))$ and not after. Suppose that $\hat{q}(T,a,s)$ does not leave $V_A(s)$ for $s \in [s_0, -\log(T-\hat{T}))$, then, $\forall y \in \mathbb{R}$, $\forall s \in [s_0, -\log(T-\hat{T}))$, $|\hat{q}(T,a,y,s)| \leq \frac{C(A)}{\sqrt{s}}$ (Cf Definition 3.2).
Since $\hat{u}(x,t) = (T-t)^{-\frac{1}{p-1}}(\hat{q}(T,a,\frac{x-a}{\sqrt{T-t}}, -\log(T-t)) + \varphi(\frac{x-a}{\sqrt{T-t}}, -\log(T-t)))$, $\limsup_{t\to\hat{T}} \|\hat{u}(t)\|_{L^\infty(\mathbb{R})} \leq C_{T,\hat{T},A} < +\infty$. This contradicts the fact that $\hat{u}(t)$ blows up at time $\hat{T}$.

Case 2: $T \leq \hat{T}$ and $(T,a) \neq (\hat{T},\hat{a})$:
$\hat{q}(T,a,s)$ is defined on $[s_0, +\infty)$. Suppose that $\hat{q}(T,a,s)$ does not leave $V_A(s)$ for $s \in [s_0, +\infty)$. Then, $\forall y \in \mathbb{R}$, $\forall s \in [s_0, +\infty)$, $|\hat{q}(T,a,y,s)| \leq \frac{C(A)}{\sqrt{s}}$ (Cf Definition 3.2). Hence, by (13),

$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(a+\sqrt{(T-t)|\log(T-t)|}z,t) - f(z)\|_{L^\infty} = 0$, and from Proposition 3.1, $u(t)$ blows up at time T at one single point; $x = a$. Since $(T,a) \neq (\hat{T},\hat{a})$, we have a contradiction. Therefore, $\hat{q}(T,a,s)$ leaves $V_A(s)$ for $s \geq s_0$.

In conclusion, we derive: $\forall (T,a) \in D\backslash\{(\hat{T},\hat{a})\}$, $\exists s_*(T,a) > s_0$ such that $\forall s \in [s_0, s_*(T,a)]$, $\hat{q}(T,a,s) \in V_A(s)$ and $\hat{q}(T,a,s_*(T,a)) \in \partial V_A(s_*(T,A))$. ($\hat{s}_*(T,a) > s_0$ since $\hat{q}(T,a,s)$ is in $V_{2\hat{A}}(s_0)$ which is strictly included in $V_A(s_0)$). If now we define $\Psi_{\hat{u}_0}$ by (59), then we see from lemma 4.2 that $Im(\Psi_{\hat{u}_0}) \subset \partial\mathcal{C}$.

Proof of iii):

Let $(T,a) \in D_5(s_0)\backslash(\hat{T},\hat{a})$. We have explicitly for $m = 0,1$:
$\hat{q}_m(T,a,s) = \int d\mu k_m(y)\chi(y,s)\hat{q}(T,a,y,s)$
$= \int d\mu k_m(y)\chi(y,s)e^{-\frac{s}{p-1}}\hat{u}(a+ye^{-s/2}, T-e^{-s}) - \int d\mu k_m(y)\chi(y,s)\varphi(y,s)$. From the continuity of $u(x,t)$ with respect to (x,t), and $ii)$ of lemma 4.2, $\hat{s}_*(T,a)$ and $\frac{\hat{s}_*(T,a)^2}{A}(q_0,q_1)(T,a,\hat{s}_*(T,a))$ are continuous with respect to (T,a).

Proof of iv):

Let $\epsilon > 0$. We now construct Γ_ϵ satisfying $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$ and $\forall (T,a) \in \Gamma_{A,s_1}$, $|(T,a) - (\hat{T},\hat{a})| \leq \epsilon$. This will be implied by the following:

Lemma 4.3 *There exists $A_6 > 0$ such that $\forall A \geq A_6$, $\exists s_6(A) > 0$ satisfying the following property:*
$\forall s_0 \geq s_6(A)$, $\exists D_6(s_0)$ neighborhood of $(\hat{T},\hat{a})$ such that $\forall \epsilon > 0$,
$\exists s_1(A,\epsilon,s_0) > s_0$, $\exists \Gamma_\epsilon$, a 1-manifold in $D_6(s_0)$ satisfying:
$\forall (T,a) \in \Gamma_\epsilon$, $|(T,a) - (\hat{T},\hat{a})| \leq \epsilon$
$\forall s \in [s_0,s_1]$, $\hat{q}(T,a,s) \in V_A(s)$,

$$(\hat{q}_0,\hat{q}_1)(T,a,s_1) \in \partial\hat{V}_A(s_1),$$

$$d(\Gamma_\epsilon, (\hat{q}_0,\hat{q}_1)(.,.,s_1), 0) = -1. \tag{60}$$

a) Proof of lemma 4.3: The proof is not difficult, but it is a bit technical. See Appendix B for more details.

b) Lemma 4.3 implies iv):

Let $A_4 = \max(A_5, A_6)$, and $A \geq A_4$. Let $s_4(A) = \max(s_5(A), s_6(A))$, and $s_0 \geq s_4(A)$. Let $D_4(s_0) = D_5(s_0) \cap D_6(s_0)$, and $\epsilon > 0$.
Then, according to the beginning of Proof of Proposition 4.2, $i)$ $ii)$ and $iii)$ hold. We take now $s_1 = s_1(A,\epsilon,s_0)$ and Γ_ϵ. By lemma 4.3, we see that $\forall (T,a) \in \Gamma_\epsilon$, $s_*(T,a) = s_1$, and $\Psi_{\hat{u}_0}(T,a) = \frac{s_1^2}{A}(\hat{q}_0,\hat{q}_1)(T,a,s_1)$. From (60), we derive, $d(\Gamma_\epsilon, \Psi_{\hat{u}_0}, 0) = -1$, which concludes the proof of Proposition 4.2.

Step 2: Behavior of $q(T,a)$ for u_0 near $\hat{u}_0$.

Now, we fix $A_0 = 1 + \sup(2\hat{A}, A_2, A_3, A_4)$, and then
$s_0 = s_0(A_0) = \sup(\hat{s}_0, s_2(A_0), s_3(A_0), s_4(A_0))$. Applying lemma 4.1 gives us $\mathcal{V}_1$, and $D_1(s_0)$. We then fix $D_0 = D_1(s_0) \cap D_4(s_0)$. Applying proposition 4.2 with s_0 and $\epsilon_0 > 0$ small enough gives us the curve $\Gamma_0 = \Gamma_{\epsilon_0}$, included in D_0. We consider now Γ_0 as fixed.

Our purpose is to show that for u_0 near $\hat{u}_0$, the behavior of $q(T,a)$ on the curve $\Gamma_0 = \Gamma(\hat{u}_0)$ is the same as $\hat{q}(T,a)$. More precisely, we have:

Proposition 4.3 (Stability result on the behavior on Γ_0, for u_0 near $\hat{u}_0$) *$\forall \epsilon > 0$, $\exists \mathcal{V}_\epsilon \subset \mathcal{V}_1$, neighborhood of $\hat{u}_0$ such that $\forall u_0 \in \mathcal{V}_\epsilon$, $\forall (T,a) \in \Gamma_0$,*
i) $q(T,a,s)$ is defined for $s \in (-\log T, s_0]$ and $q(T,a,s_0) \in V_{A_0}(s_0)$,
ii) $\exists s_*(T,a) > s_0$ such that $\forall s \in [s_0, s_*(T,a)]$, $q(T,a,s) \in V_{A_0}(s)$, and $(q_0,q_1)(T,a,s_*(T,a)) \in \partial \hat{V}_{A_0}(s_*(T,a))$. Then we can define

$$\begin{aligned} \Psi_{u_0} : \gamma &\longrightarrow \partial \mathcal{C} \\ (T,a) &\longrightarrow \frac{s_*(T,a)^2}{A_0}(q_0,q_1)(T,a,s_*(T,a)) \end{aligned} \tag{61}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Moreover,
iii) Ψ_{u_0} is a continuous mapping from Γ_0 to $\partial\mathcal{C}$,
iv) $\|\Psi_{u_0|\Gamma_0} - \Psi_{\hat{u}_0|\Gamma_0}\|_{L^\infty(\Gamma_0)} \le \epsilon$

Proof:
We first show a local result, then by compactness arguments we conclude the proof. We claim the following:

Lemma 4.4 (Punctual stability on Γ_0) *$\forall \epsilon > 0$, $\forall (T,a) \in \Gamma_0$, $\exists D_{\epsilon,T,a}$ neighborhood of (T,a) in D_0, $\exists \mathcal{V}_{\epsilon,T,a}$ neighborhood of $\hat{u}_0$ in $\mathcal{V}$ such that:*
$\forall (T',a') \in D_{\epsilon,T,a}$, $\forall u_0 \in \mathcal{V}_{\epsilon,T,a}$,
i) $q(T',a',s)$ is defined (at least) for $s \in (-\log T, s_0]$ and $q(T',a',s_0) \in V_{A_0}(s_0)$,
ii)$\exists s_*(T',a') > s_0$ such that $\forall s \in [s_0, s_*(T',a')]$, $q(T',a',s) \in V_{A_0}(s)$, and $(q_0,q_1)(T',a',s_*(T',a')) \in \partial \hat{V}_{A_0}(s_*(T',a'))$.
Moreover,

$$\left|\frac{s_*(T',a')^2}{A_0}(q_0,q_1)(T',a',s_*(T',a')) - \frac{s_*(T',a')^2}{A_0}(\hat{q}_0,\hat{q}_1)(T',a',\hat{s}_*(T',a'))\right| \le \epsilon. \tag{62}$$

We remark that Proposition 4.3 follows from lemma 4.4. Indeed, for $\epsilon > 0$, from lemma we write:

$$\Gamma_0 \subset \cup_{(T,a)\in\Gamma_0} D_{\epsilon,T,a},$$

and using the compactness of Γ_0, we have the conclusion.

Proof of Lemma 4.4
We have explicitly for $u_0 \in H$, $s \in (-\log T, -\log(T-\hat{T}))$ if $T > \hat{T}$, otherwise $s \in (-\log T, +\infty)$, and $m = 0,1$
$q_{m,T,a}(s) = \int d\mu k_m(y)\chi(y,s)q(T,a,y,s)$
$= \int d\mu k_m(y)\chi(y,s)e^{-\frac{s}{p-1}}u(a+ye^{-s/2}, T-e^{-s}) - \int d\mu k_m(y)\chi(y,s)\varphi(y,s)$. Therefore, using the continuity of $u(x,t)$ with respect to (u_0,x,t), $(q_0,q_1)(T,a,s)$ is a continuous function of (u_0,T,a,s). Using this fact and the transversality of $(\hat{q}_0,\hat{q}_1)(T,a,\hat{s}_*(T,a))$ on $\hat{V}_{A_0}(s_*(T,a))$ (lemma 4.2 *ii*)), *i*) and *ii*) follow then easily.
This concludes the proof of Proposition 4.3.

Step 3: The conclusion of the proof

From continuity properties of the topological degree, there exists $\epsilon_1 > 0$ such that $\forall \Psi \in \mathcal{C}(\Gamma_0, \mathbb{R}^2)$ satisfying $\|\Psi - \Psi_{\hat{u}_0}\|_{L^\infty(\Gamma_0)} \le \epsilon_1$, we have $d(\Gamma_0, \Psi, 0) = -1$. Applying Proposition 4.3, with $\epsilon = \epsilon_1$, we have $\forall u_0 \in \mathcal{V}_{\epsilon_1}$, $d(\Gamma_0, \Psi_{u_0}, 0) = -1$.

We claim that the conclusion of Proposition 4.1 follows with A_0, s_0, D_0 and $\mathcal{V}_0 = \mathcal{V}_{\epsilon_1}$. Indeed, by contradiction as in section 3: suppose that for $u_0 \in \mathcal{V}_0$, we have $\forall (T,a) \in D_0$, there exists $s \geq s_0$, $q(T,a,s) \notin V_{A_0}(s)$. Let $s_*(T,a)$ be the infimum of all these s. We now remark that Ψ_{u_0} is defined on D_0 (lemma 4.1 and lemma 4.2). Ψ_{u_0} is continuous from D_0 to $\partial\mathcal{C}$ (see proof of Proposition 4.2 *iii*), and $d(\Gamma_0, \Psi_{u_0}, 0) = 0$, which is a contradiction. Hence Proposition 4.1 is proved, which concludes the proof of Theorem 2.

4.2 Case $N \geq 2$:

Let us consider $\hat{u}_0$ an initial data in H, constructed in Theorem 1. Let $\hat{u}(t)$ be the solution of equation (1):

$$u_t = \Delta u + |u|^{p-1}u, u(0) = \hat{u}_0.$$

Let $\hat{T}$ be its blow-up time and $\hat{a}$ be its blow-up point.

Although the proof of Theorem 1 was given in 1 dimension, we know that there exists $\hat{A} > 0$, $\hat{s}_0 > \log \hat{T}$ such that $\forall s \geq \hat{s}_0$, $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$, where:
- $\hat{q}_{\hat{T},\hat{a}}$ is defined in (13) by:

$$q_{\hat{T},\hat{a}}(y,s) = e^{-\frac{s}{p-1}}\hat{u}(\hat{a} + ye^{-\frac{s}{2}}, \hat{T} - e^{-s}) - [\frac{N\kappa}{2ps} + (p - 1 + \frac{(p-1)^2}{4ps}|y|^2)^{-\frac{1}{p-1}}],$$

and
-Definitions 3.1 and 3.2 are still good to define $V_{\hat{A}}(s)$, if we understand $q_m(s)$ to be a vector valued function, as defined in section 2 (see (27) and (28)), and $|q_m(s)|$ to be the supremum of of all coordinates of $q_m(s)$. (By the same way, the definition of $\hat{V}_A(s)$ given in 3.3 is good here).

With these adaptations, our purpose is summarized in the following Proposition, analogous to Proposition 4.1:

Proposition 4.4 (Reduction) *There exist $A_0 > 0$, $s_0 > 0$, D_0 neighborhood of $(\hat{T}, \hat{a})$ in $\mathbb{R}^{1+N}$, and $\mathcal{V}_0$ neighborhood of $\hat{u}_0$ in H with the following property: $\forall u_0 \in \mathcal{V}_0, \exists (T,a) \in D_0$ such that $\forall s \geq s_0$, $q_{T,a}(s) \in V_{A_0}(s)$, where $q_{T,a}$ is defined by (13), and $u(t)$ is the solution of equation (1) with initial data $u(0) = u_0$.*

Indeed, once this proposition is proved, from (3), (13) and definitions 3.1, 3.2, we have:

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z\ , t) = f(z)$$

uniformly in $z \in \mathbb{R}^N$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}.$$

Proposition 3.1 (which is true in N dimensions) applied to $u(x-a,t)$ then shows directly that $u(t)$ blows-up at time T at one single point: $x = a$.

Formally, the proof in the case $N \geq 2$ and in the case $N = 1$ have exactly the same steps with the same statements of Propositions and lemmas, under the following obvious changes:

-$(\hat{T},\hat{a})$, (T,a) and (T',a') are in $\mathbb{R}^{1+N}$, and every neighborhood of such a point is a neighborhood in $\mathbb{R}^{1+N}$.

-In Part 2, $\mathcal{C}$ denotes the unit (1+N)-cube of $\mathbb{R}^{1+N}$, Γ (and Γ_ϵ, Γ_0,...) is a Lipschitz N-submanifold of $\mathbb{R}^{1+N}$, forming the boundary of a bounded connected Lipschitz open set of $\mathbb{R}^{1+N}$, and all introduced topological degrees different from zero are equal to $(-1)^N$.

Moreover, the proofs can be adapted without difficulty to the case $N \geq 2$, even:
-the proof of Proposition 4.2, which relays on results of section 3 (subsection 3.3 and lemma 3.2) that are true in N dimensions (In particular, the lemma 3.5 of Bricmont and Kupiainen, with the adaptation $\mathbb{R} \to \mathbb{R}^N$).
-the construction of Γ_ϵ given in Appendix B can be simply adapted to the case $N \geq 2$.

A Proof of lemma 3.5

In this appendix, we prove lemma 3.5. Equation (15) has been studied in [3], hence, our analysis will be very close to [3] (the proof is essentially the same as in [3]). Lemma 3.5 relays mainly on the understanding of the behavior of the kernel $K(s,\sigma,y,x)$ (see (20)). This behavior follows from a perturbation method around $e^{(s-\sigma)\mathcal{L}}(y,x)$.

Step 1: Perturbation formula for $K(s,\sigma,y,x)$

Since $\mathcal{L}$ is conjugated to the harmonic oscillator $e^{-x^2/8}\mathcal{L}e^{x^2/8} = \partial^2 - \frac{x^2}{16} + \frac{1}{4} + 1$, we use the definition (20) of K and give a Feynman-Kac representation for K:

$$K(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma}V(\omega(\tau),\sigma+\tau)d\tau} \tag{63}$$

where $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0,s-\sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s-\sigma) = y$, i.e. the Gaussian probability measure with covariance kernel $\Gamma(\tau,\tau') = \omega_0(\tau)\omega_0(\tau')$

$$+2(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)-\tau'+\tau|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau'-\tau|}, \tag{64}$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau) = \omega_0(\tau)$ with
$\omega_0(\tau) = (\sinh\frac{s-\sigma}{2})^{-1}(y\sinh\frac{\tau}{2} + x\sinh\frac{s-\sigma-\tau}{2})$.

We have in addition

$$e^{\theta\mathcal{L}}(y,x) = \frac{e^\theta}{\sqrt{4\pi(1-e^{-\theta})}}\exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}].$$

Now, we derive from (63) a simplified expression for $K(s,\sigma,y,x)$ considered as a perturbation of $e^{(s-\sigma)\mathcal{L}}(y,x)$. In order to simplify the notation, we write from now on (ψ,φ) for $\int d\mu(y)\psi(y)\varphi(y)$.

Lemma A.1 (Bricmont-Kupiainen) *$\forall s \geq \sigma \geq 1$ with $s \leq 2\sigma$, the kernel $K(s,\sigma,y,x)$ satisfies*

$$K(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)(1+\frac{1}{s}P_1(s,\sigma,y,x)+P_2(s,\sigma,y,x))$$

where P_1 is a polynomial

$$P_1(s,\sigma,y,x) = \sum_{m,n\geq 0, m+n\leq 2} p_{m,n}(s,\sigma)y^m x^n$$

with $|p_{m,n}(s,\sigma)| \leq C(s-\sigma)$ and

$$|P_2(s,\sigma,y,x)| \leq C(s-\sigma)(1+s-\sigma)s^{-2}(1+|y|+|x|)^4.$$

Moreover, $|(k_2,(K(s,\sigma)-(\sigma s^{-1})^2)h_2)| \leq C(s-\sigma)(1+s-\sigma)s^{-2}$.

Proof: See lemma 5 in [3].

Step 2: Conclusion of the proof of lemma 3.5

Proof of a): From (16), it follows easily that $V(y,s) \leq Cs^{-1}$. Using this estimate and (63), we write:
$|K(s,\tau,y,x)| \leq e^{(s-\tau)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\tau}(\omega)e^{\int_0^{s-\tau}C(\tau+t)^{-1}dt}$
$\leq e^{(s-\tau)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\tau}(\omega)(s\tau^{-1})^C \leq Ce^{(s-\tau)\mathcal{L}}(y,x)$ since $s \leq 2\tau$ and $d\mu_{yx}^{s-\tau}$ is a probability.

Proof of c): See lemma 2 in [3].

Proof of b): We consider $A' > 0$, $A'' > 0$, $A''' > 0$ and $\rho^* > 0$. Let $s_0 \geq \rho^*$, $\sigma \geq s_0$ and $q(\sigma)$ satisfying (46). We want to estimate some components of $\alpha(y,s) = K(s,\sigma)q(\sigma)$ (see (47)) for each $s \in [\sigma, \sigma+\rho^*]$.

Since $\sigma \geq s_0 \geq \rho^*$, we have: $\forall \tau \in [\sigma,s]$, $\tau \leq s \leq 2\tau$. Therefore, up to a multiplying constant, any power of any $\tau \in [\sigma,s]$ will be bounded systematically by the same power of s during the proof.

i) Estimate of $\alpha_2(s)$:
$\alpha_2(s) = (k_2, \chi(.,s)K(s,\sigma)q(\sigma))$
$= \sigma^2 s^{-2} q_2(\sigma) + (k_2, (\chi(.,s)-\chi(.,\sigma))\sigma^2 s^{-2} q(\sigma))$
$+(k_2, \chi(.,s)(K(s,\sigma)-\sigma^2 s^{-2})q(\sigma))$.

From (46), (21) and (26), we have $|\sigma^2 s^{-2} q_2(\sigma)| \leq A'' s^{-2}\log\sigma$ and
$|(k_2,(\chi(.,s)-\chi(.,\sigma))\sigma^2 s^{-2}q(\sigma))| \leq Ce^{-C\sigma}\sigma^{-3/2}(s-\sigma)\sigma^2 s^{-2}\frac{\max(A'',A''')}{\sqrt{\sigma}}$
$\leq CA'(s-\sigma)s^{-3}$ for $\sigma \geq s_0 \geq s_1(A',A'',A''',\rho^*)$.

We write $(k_2,\chi(.,s)(K(s,\sigma)-\sigma^2 s^{-2})q(\sigma))$ as $\sum_{r=0}^{2} b_r + b_- + b_e$ where
$b_r = (k_2,\chi(.,s)(K(s,\sigma)-\sigma^2 s^{-2})h_r)q_r(\sigma)$,
$b_- = (k_2,\chi(.,s)(K(s,\sigma)-\sigma^2 s^{-2})q_-(\sigma))$ and
$b_e = (k_2,\chi(.,s)(K(s,\sigma)-\sigma^2 s^{-2})q_e(\sigma))$.

For $r=0$ or 1, we use lemma A.1, corollary 3.1, (21), (46), the fact that $e^{(s-\sigma)\mathcal{L}}h_r = e^{(1-r/2)(s-\sigma)}h_r$ and $(k_2,h_r)=0$, and derive $|b_r| =$
$|(k_2,\chi(.,s)(K(s,\sigma)-e^{(s-\sigma)\mathcal{L}})h_r)q_r(\sigma) + (k_2,\chi(.,s)(e^{(s-\sigma)\mathcal{L}}-\sigma^2 s^{-2})h_r)q_r(\sigma)|$
$\leq CA'(s-\sigma)s^{-3} + Ce^{-Cs}(s-\sigma) \leq CA'(s-\sigma)s^{-3} \leq CA'(s-\sigma)s^{-3}$.

We have by lemma A.1 and the same arguments $|b_2| = |(k_2, (K(s,\sigma) - \sigma^2 s^{-2})h_2)q_2(\sigma) + (k_2, (-1+\chi(.,s))(K(s,\sigma) - \sigma^2 s^{-2})h_2)q_2(\sigma)| \leq C(s-\sigma)(1+s-\sigma)s^{-2}A''s^{-2}\log s + Ce^{-Cs}(s-\sigma) \leq CA'(s-\sigma)s^{-3}$ if $\sigma \geq s_0 \geq s_2(A', A'', \rho^*)$.

b_- can be treated exactly as b_0, it is bounded by $C(s-\sigma)A'''s^{-3}$.

Since $K(s,\sigma) - \sigma^2 s^{-2} = K(s,\sigma) - e^{(s-\sigma)\mathcal{L}} + (e^{(s-\sigma)\mathcal{L}} - 1) + (1 - \sigma^2 s^{-2})$, we write $b_e = b_{e,1} + b_{e,2} + b_{e,3}$ with $b_{e,1} = (k_2, \chi(.,s)(K(s,\sigma) - e^{(s-\sigma)\mathcal{L}})q_e(\sigma))$, $b_{e,2} = (k_2, \chi(.,s)\int_0^{s-\sigma} d\tau \mathcal{L}e^{\tau\mathcal{L}}q_e(\sigma))$, $b_{e,3} = (k_2, \chi(.,s)(1-\sigma^2 s^{-2})q_e(\sigma))$.

From (46), we bound $b_{e,3}$ by $C(s-\sigma)s^{-1}A''\sigma^{-1/2}e^{-C\sigma} \leq C(s-\sigma)A's^{-3}$ if $\sigma \geq s_0 \geq s_3(A', A'', \rho^*)$. Since $\mathcal{L}$ is self-adjoint, $|b_{e,2}| \leq \int \frac{e^{-y^2/4}}{\sqrt{4\pi}} dy \mathcal{L}(k_2\chi(.,s))(y) \int_0^{s-\sigma} d\tau \int dx \frac{e^{(s-\sigma)}}{\sqrt{4\pi(1-e^{-1})}} \exp[-\frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]A''\sigma^{-1/2}$.

Now, we have $e^{\frac{1}{2}[-\frac{y^2}{4} - \frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]} \leq e^{-C(K_0)s} \leq e^{-2s}$, for $|y| \leq 2K_0\sqrt{s}$ and $|x| \geq K_0\sqrt{\sigma}$ (if K_0 is big enough and $s_0 \geq \rho^*$). Hence,
$|b_{e,2}| \leq CA''s^{-1/2} \int e^{-y^2/8} dy \int_0^{s-\sigma} d\tau \int dx \frac{e^{-s}}{\sqrt{4\pi(1-e^{-1})}} \exp[-\frac{1}{2}\frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}]$
$\leq CA''s^{-1/2}(s-\sigma)e^{-s} \leq CA'(s-\sigma)s^{-3}$ if $\sigma \geq s_0 \geq s_4(A', A'', \rho^*)$.

Using these techniques and lemma A.1 we bound $b_{e,1}$ in the same way.

Adding all these bounds yields the bound for $|\alpha_2(s)|$.

ii) Estimate of $\alpha_-(y,s)$:

By definition, $\alpha_-(y,s)$

$$= P_-(\chi(.,s)K(s,\sigma)q(\sigma)) = P_-(\chi(.,s)K(s,\sigma)q_-(\sigma))$$
$$+\sum_{r=0}^{2} q_r(\sigma)P_-(\chi(.,s)K(s,\sigma)h_r) + P_-(\chi(.,s)K(s,\sigma)q_e(\sigma)) \tag{65}$$

where P_- is the $L^2(\mathbb{R}, d\mu)$ projector on the negative subspace of $\mathcal{L}$ (see subsection 2.2). In order to bound the first term, we proceed as in [3]

$$K(s,\sigma)q_-(\sigma) = \int dx e^{x^2/4} K(s,\sigma,.,x)f(x) \tag{66}$$

where $f(x) = e^{-x^2/4}q_-(x,\sigma)$. From Step 1, we have
$e^{x^2/4}K(s,\sigma,y,x) = N(y,x)E(y,x)$ with

$$N(y,x) = [4\pi(1-e^{-(s-\sigma)}]^{-1/2}e^{s-\sigma}e^{x^2/4}e^{-\frac{(y-e^{-(s-\sigma)/2}x)^2}{4(1-e^{-(s-\sigma)})}} \tag{67}$$

and $E(y,x) = \int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma} V(\omega(\tau),\sigma+\tau)d\tau}$. Let $f^0 = f$ and for $m \geq 1$, $f^{(-m-1)}(y) = \int_{-\infty}^{y} dx f^{(-m)}(x)$. From (46) and the following lemma, we can bound $f^{(-m)}$:

Lemma A.2 $|f^{(-m)}(y)| \leq CA'''s^{-2}(1+|y|^3)^{3-m}e^{-y^2/4}$.

Proof: See lemma 6 in [3].

By integrating by parts, we rewrite (66) as:

$$(K(s,\sigma)q_-(\sigma))(y) = \sum_{r=0}^{2}(-1)^{r+1}\int \partial_x^r N(y,x)\partial_x E(y,x)f^{(-r-1)}(x)dx$$
$$- \int \partial_x^3 N(y,x)E(y,x)f^{(-3)}(x)dx. \tag{68}$$

From (67), we get for $s-\sigma \geq 1$ and $r \in \{0,1,2,3\}$
$|\partial_x^r N(y,x)| \leq Ce^{-\frac{r(s-\sigma)}{2}}(1+|y|+|x|)^r e^{x^2/4} e^{(s-\sigma)\mathcal{L}}(y,x)$.

Using the integration by parts formula for Gaussian measures (see [11]), we have:

$$\begin{aligned}\partial_x E(y,x) &= \frac{1}{2}\int_0^{s-\sigma}\int_0^{s-\sigma} d\tau d\tau' \partial_x \Gamma(\tau,\tau') \int d\mu_{yx}^{s-\sigma}(\omega) V'(\omega(\tau),\sigma+\tau) \\ (69) \quad & V'(\omega(\tau'),\sigma+\tau') e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')} \\ &+ \frac{1}{2}\int_0^{s-\sigma} d\tau \partial_x \Gamma(\tau,\tau) \int d\mu_{yx}^{s-\sigma}(\omega) V''(\omega(\tau),\sigma+\tau) e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')}.\end{aligned}$$

By (16), we have $V(y,s) \leq Cs^{-1}$ and $|\frac{d^n V}{dy^n}| \leq Cs^{-n/2}$ for $n = 0,1,2$. Combining this with (64) and using $s \leq 2\sigma$ we have
$\int d\mu_{yx}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')} \leq C$ and $|\partial_x E(y,x)| \leq Cs^{-1}(s-\sigma)(1+s-\sigma)(|y|+|x|)$.

Using (46), (68) and all these bounds, we get
$|(K(s,\sigma)q_-(\sigma))(y)| \leq CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $\sigma \geq s_0 \geq s_5(\rho^*)$ and $s-\sigma \geq 1$. This yields $|(P_-\chi(.,s)K(s,\sigma)q_-(\sigma))(y)| \leq CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $s-\sigma \geq 1$. For $s-\sigma \leq 1$, we use directly lemma A.1, corollary 3.1, (46) and $C \leq e^{-(s-\sigma)/2}$ to get the same estimate.

Now, we consider the second term in (65) ($r = 0,1,2$). From corollary 3.1, lemma A.1, and the fact that $|y| \leq 2K_0 s^{1/2}$, we obtain:

$$\begin{aligned}& |q_r(\sigma)(\chi(.,s)K(s,\sigma)h_r)(y) - q_r(\sigma)e^{(s-\sigma)(1-r/2)}(\chi(.,s)h_r)(y)| \\ (70) \quad & \leq C\max(A',A'')s^{-3+1/2}\log s.(s-\sigma)(1+s-\sigma)e^{s-\sigma}(1+|y|^3)\end{aligned}$$

Hence $P_-\{q_r(\sigma)(\chi(.,s)K(s,\sigma)h_r)(y) - q_r(\sigma)e^{(s-\sigma)(1-r/2)}(\chi(.,s)h_r)(y)\}$ satisfies the same bound. Since $P_-h_r = 0$ and $|(1-\chi(.,s))h_r| \leq Cs^{-1/2}(1+|y|^3)$, we can bound $q_r(\sigma)e^{(s-\sigma)(1-r/2)}P_-(\chi(.,s)h_r)$ by (70). Hence, the second term of (65) is bounded by $CA'''s^{-2}e^{-(s-\sigma)/2}(1+|y|^3)$ if $\sigma \geq s_0 \geq s_6(A',A'',A''',\rho^*)$.

For the last term in (65), we use (46) and a) of lemma 3.5 to get

$$\begin{aligned}&\|(1+|y|^3)^{-1}\chi(.,s)K(s,\sigma)q_e(\sigma)\|_{L^\infty} \leq CA''e^{s-\sigma}s^{-1/2}\sup_{y,x}(1+|y|^3)^{-1} \\ &. \exp[-\frac{1}{2}\frac{(x-ye^{-(s-\sigma)/2})^2}{4(1-e^{-(s-\sigma)})}]\chi(y,\sigma+(s-\sigma))(1-\chi(x,\sigma)) \\ &\leq \begin{cases} CA''s^{-2} & s-\sigma \leq t_0 \\ e^{-s} & s-\sigma \geq t_0 \end{cases}\end{aligned}$$

for a suitable constant t_0. This yields a bound on the last term in (65) which can be written as $CA''e^{-(s-\sigma)^2}s^{-2}(1+|y|^3)$ for $\sigma \geq s_0$ large enough.

Hence, combining all bounds for terms in (65), we have

$$|\alpha_-(y,s)| \leq Cs^{-2}(A'''e^{-(s-\sigma)/2} + A''e^{-(s-\sigma)^2})(1+|y|^3).$$

Estimate of $\alpha_e(y,s)$:

We write $\alpha_e(y,s) = (1-\chi(y,s))K(s,\sigma)q(\sigma) = (1-\chi(y,s))K(s,\sigma)(q_b(\sigma) + q_e(\sigma))$. From (46) and corollary 3.1, we have $|q_b(y,s)| \leq CA'''\sigma^{-1/2}$ and $\|(1-$

$\chi(y,s))K(s,\sigma)q_b(\sigma)\|_{L^\infty} \leq A'''e^{s-\sigma}s^{-1/2}$ if $\sigma \geq s_0 \geq s_7(A',A'',A''')$. Using (46) and the following lemma from [3]:

Lemma A.3 $\|K(s,\sigma)(1-\chi(\sigma))\|_{L^\infty} \leq Ce^{-(s-\sigma)/p}$

we have $\|(1-\chi(y,s))K(s,\sigma)q_e(\sigma)\|_{L^\infty} \leq A''e^{-(s-\sigma)/p}s^{-1/2}$, which yields the conclusion.

This concludes the proof of lemma 3.5.

B Proof of lemma 4.3

Let us recall lemma 4.3:

Lemma B.1 *There exists $A_6 > 0$ such that $\forall A \geq A_6$, $\exists s_6(A) > 0$ satisfying the following property:*
$\forall s_0 \geq s_6(A)$, $\exists D_6(s_0)$ neighborhood of $(\hat{T},\hat{a})$ such that $\forall \epsilon > 0$,
$\exists s_1(A,\epsilon,s_0) > s_0$, $\exists \Gamma_\epsilon$, a 1-manifold in $D_6(s_0)$ satisfying:
$\forall (T,a) \in \Gamma_\epsilon$, $|(T,a)-(\hat{T},\hat{a})| \leq \epsilon$
$\forall s \in [s_0,s_1]$, $\hat{q}(T,a,s) \in V_A(s)$,

$$(\hat{q}_0,\hat{q}_1)(T,a,s_1) \in \partial\hat{V}_A(s_1), \tag{71}$$

$$d(\Gamma_\epsilon,(\hat{q}_0,\hat{q}_1)(.,.,s_1),0) = -1. \tag{72}$$

In this lemma, we want to control the evolution of $\hat{q}(T,a,s)$ in $V_A(s)$, for (T,a) close to $(\hat{T},\hat{a})$. Hence, in a first step, we use $\hat{q}_{\hat{T},\hat{a}}(s) \in V_{\hat{A}}(s)$ $\forall s \geq \hat{s}_0$, to give estimates on different components of $\hat{q}_{T,a}(s)$, for (T,a) near $(\hat{T},\hat{a})$. From these estimates, we introduce a function $(\tilde{q}_0,\tilde{q}_1)(T,a,s)$ close to $(\hat{q}_0,\hat{q}_1)(T,a,s)$, but much more simple, and show that $(\tilde{q}_0,\tilde{q}_1)$ satisfies properties analogous to (71) and (72). Therefore, we extend this result to $(\hat{q}_0,\hat{q}_1)$, by continuity, and then finish the proof of lemma 4.3.

Step 1: Asymptotic development of $\hat{q}(T,a)$ for (T,a) near $(\hat{T},\hat{a})$
Applying (13) and (3), one time to $(\hat{T},\hat{a})$ and one time to (T,a), we write:

$$\begin{aligned} \hat{q}(T,a,y,s) &= \{(1-\tau)^{-\frac{1}{p-1}}\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}},s-\log(1-\tau))\} \\ &+ \{(1-\tau)^{-\frac{1}{p-1}}(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} \\ &- (p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}\} \\ &+ \{(1-\tau)^{-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))}-\frac{\kappa}{2ps}\}, \end{aligned} \tag{73}$$

with $\tau = (T-\hat{T})e^s$, and $\alpha = (a-\hat{a})e^{s/2}$. Now, we use $\hat{q}(\hat{T},\hat{a},s) \in V_{\hat{A}}(s)$ for $s \geq \hat{s}_0$, to give a development of $\hat{q}_{T,a}(y,s)$, when $|\tau| \leq \frac{1}{2}$, and $|\alpha| \leq \frac{1}{2}$.

Lemma B.2 (development of $\hat q(T,a)$ near $(\hat T,\hat a)$) *There exists $s_7>0$ such that $\forall s\geq s_7$, $\forall (T,a)\in\mathbb{R}^2$ satisfying $|(T-\hat T)e^s|\leq \frac12$ and $|(a-\hat a)e^{\frac s2}|\leq\frac12$, we have:*

$$\hat q_0(T,a,s) = \tilde q_0(T,a,s)+O\left(\frac{\log s}{s^{5/2}}+\frac{\tau}{\sqrt s}+\tau^2+\alpha^2\frac1s\right) \tag{74}$$

$$\hat q_1(T,a,s) = \tilde q_1(T,a,s)+O\left(\frac{\alpha\log s}{s^2}+\frac{\alpha^2}{s}+\frac{\tau}{s}+\frac{\log s}{s^3}\right)$$

$$\frac{\partial\hat q_0}{\partial T}(T,a,s) = \frac{\partial\tilde q_0}{\partial T}(T,a,s)+e^s(O(\tau+s^{-1/2})), \tag{75}$$

$$\frac{\partial\hat q_0}{\partial a}(T,a,s) = \frac{\partial\tilde q_0}{\partial a}(T,a,s)+e^{s/2}O\left(\frac{\log s}{s^2}+\frac{|\alpha|}{s}\right), \tag{76}$$

$$\frac{\partial\hat q_1}{\partial T}(T,a,s) = \frac{\partial\tilde q_1}{\partial T}(T,a,s)+e^sO\left(\frac{1}{\sqrt s}\right), \tag{77}$$

$$\frac{\partial\hat q_1}{\partial a}(T,a,s) = \frac{\partial\tilde q_1}{\partial a}(T,a,s)+e^{s/2}O\left(\frac{|\tau|}{s}+\frac{1}{s^2}+\frac{|\alpha|}{s}\right) \tag{78}$$

with

$$\tilde q_0(T,a,s) = -\frac{5\kappa}{8ps^2}+\tau\frac{\kappa}{p-1} \tag{79}$$

$$\tilde q_1(T,a,s) = -\frac{\alpha}{s}\frac{\kappa}{2p},$$

and $\tau=(T-\hat T)e^s$ and $\alpha=(a-\hat a)e^{\frac s2}$.
Moreover,

$$|\hat q_2(T,a,s)| \leq C\frac{\log s}{s^2}+C\frac{|\tau|}{s}+C\tau^2$$

$$|\hat q_-(T,a,y,s)| \leq C(1+|y|^3)\left(\frac{1}{s^2}+\frac{|\tau|+|\alpha|}{s^{3/2}}\right)$$

$$|\hat q_e(T,a,y,s)| \leq \frac{C}{\sqrt s}.$$

Proof of lemma B.2:
The idea is simple: for $s\geq\hat s_0$, , we try to express each component of $\hat q(T,a)$ in terms of the corresponding component of $\hat q(\hat T,\hat a)$, and bound the residual terms using $\hat q(\hat T,\hat a,s)\in V_{\hat A}(s)$ and other estimates that follow from.
Hence, we first give various estimates following from $\hat q(\hat T,\hat a,s)\in V_{\hat A}(s)$, and then , we prove only some of the estimates in lemma B.2, since the other estimates can be obtained in the same way.

i) We write the estimates following from $\hat q(\hat T,\hat a,s)\in V_{\hat A}(s)$.

Lemma B.3 (Consequences of $\hat q(\hat T,\hat a,s)\in V_{\hat A}(s)$) *$\exists s_{16}>0$, $\forall s\geq s_{16}$,*

$$|\hat q(\hat T,\hat a,y,s)|\leq\frac{C}{\sqrt s}, \tag{80}$$

$$|\hat q_b(\hat T,\hat a,y,s)|\leq\frac{C\log s}{s^2}(1+|y|^3), \tag{81}$$

$$\hat q_0(\hat T,\hat a,s)=-\frac{5\kappa}{8ps^2}+o\left(\frac{1}{s^2}\right),\left|\frac{\partial\hat q_0}{\partial s}(\hat T,\hat a,s)\right|\leq\frac{C}{s^2}, \tag{82}$$

$$(83)\qquad |\hat{q}_1(\hat{T},\hat{a},s)| \le C\frac{\log s}{s^3},$$

$$(84)\qquad |\frac{\partial \hat{q}}{\partial s}(\hat{T},\hat{a},y,s)| \le C\frac{1+|y|}{\sqrt{s}},$$

Proof of lemma B.3:

(80) and (81) follow directly from Definition 3.2.

After some simple calculations, we show that $\int d\mu\chi(y,s)R(y,s) = \frac{5\kappa}{8ps^2} + O(s^{-3})$. As in the proof of lemma 3.2, we write the equation satisfied by $q_0(s)$:

$$\frac{d\hat{q}_0}{ds}(\hat{T},\hat{a},s) = \hat{q}_0(\hat{T},\hat{a},s) + \frac{5\kappa}{8ps^2} + O(\frac{\log s}{s^3}),$$

which implies (82).

By the same way, we write:

$$\frac{d\hat{q}_1}{ds}(\hat{T},\hat{a},s) = \frac{1}{2}\hat{q}_1(\hat{T},\hat{a},s) + O(\frac{\log s}{s^3}),$$

which yields (83).

From (80), we derive that $r = \frac{\partial q}{\partial s}$ satisfies

$$\frac{\partial r}{\partial s} = \frac{\partial^2 r}{\partial y^2} - \frac{1}{2}y\frac{\partial r}{\partial y} + A(y,s)r + D(y,s),$$

with $|A(y,s)| \le C$ and, if $p \ge \frac{3}{2}$ $|D(y,s)| \le \frac{C}{s}$, otherwise, $|D(y,s)| \le \frac{C}{s^{p-\frac{1}{2}}}$. By parabolic regularity, (84) follows.

ii) Proof of some estimates in lemma B.2: (74) and (75)
(The other estimates follow from similar techniques).

From (73), we have: $\hat{q}_0(T,a,s) = I_1 + I_2 + I_3$, with
$I_1 = (1-\tau)^{-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}},s-\log(1-\tau))$,
$I_2 = (1-\tau)^{-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}}$
$-\int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}$
$I_3 = \int d\mu(y)\chi(y,s)\{(1-\tau)^{-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))} - \frac{\kappa}{2ps}\}$.

-I_3*:* We have easily: $|I_3| \le C|\tau|s^{-1}$.

-I_2*:* Since all quantities appearing in I_2 are bounded, we can write:
$I_2 = O(e^{-s}) + \int d\mu(y)\{(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} - (p-1+\frac{(p-1)^2y^2}{4ps})^{-\frac{1}{p-1}}\}$
$+\frac{\tau}{p-1}\int d\mu(y)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} + O(\tau^2)$,
$= O(e^{-s}) + O(\tau^2)$
$+\int d\mu(y)\{\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))} - \frac{(p-1)^2y^2}{4ps}\}\frac{-1}{p-1}(p-1+\frac{(p-1)^2y^2}{4ps})^{-1-\frac{1}{p-1}}$
$+O(\int d\mu(y)\{\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))} - \frac{(p-1)^2y^2}{4ps}\}^2)$
$+\frac{\tau\kappa}{p-1} + \frac{\tau}{p-1}\{\int d\mu(y)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} - \int d\mu(y)\kappa\}$, hence,
$|I_2 - \frac{\tau\kappa}{p-1}| \le Ce^{-s} + C\tau^2 + C|\tau|s^{-1} + C\alpha^2s^{-1} + C\tau^2s^{-2} + C\alpha^2s^{-2} + C\alpha^4s^{-2} + C|\tau|s^{-1}$. Therefore,
$|I_2 - \frac{\tau\kappa}{p-1}| \le Ce^{-s} + C\tau^2 + C|\tau|s^{-1} + C\alpha^2s^{-1}$.

-I_1*:* Using (80), we write:
$I_1 = O(\tau s^{-1/2}) + \int d\mu(y)\chi(y,s)\hat{q}(\hat{T},\hat{a},\frac{y+\alpha}{\sqrt{1-\tau}},s-\log(1-\tau))$. If we introduce a

new integration variable: $z = \frac{y+\alpha}{\sqrt{1-\tau}}$, we obtain: $I_1 = O(\tau s^{-1/2}) + L_1 + L_2$ with
$L_1 = \int \chi(z, s-\log(1-\tau))\hat{q}(\hat{T}, \hat{a}, z, s-\log(1-\tau))\frac{\exp(-\frac{(z\sqrt{1-\tau}-\alpha)^2}{4})}{4\pi}dz$, and
$L_2 = \int \{\chi(z\sqrt{1-\tau}-\alpha, s) - \chi(z, s-\log(1-\tau))\}\hat{q}(\hat{T}, \hat{a}, z, s-\log(1-\tau))$
$\frac{\exp(-\frac{(z\sqrt{1-\tau}-\alpha)^2}{4})}{4\pi}dz$.
$L_1 = \int \chi(z, s-\log(1-\tau))\hat{q}(\hat{T}, \hat{a}, z, s-\log(1-\tau))\frac{\exp(-\frac{z^2}{4})}{4\pi}\exp(\frac{\tau z^2}{4})$
$\exp(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4})dz$
$= O(\tau s^{-1/2}) + \int \chi(z, s-\log(1-\tau))\hat{q}(\hat{T}, \hat{a}, z, s-\log(1-\tau))$
$\frac{\exp(-\frac{z^2}{4})}{4\pi}\{1 + \frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4} + \frac{1}{2}(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4})^2\int_0^1 \exp(\xi(\frac{2\alpha z\sqrt{1-\tau}-\alpha^2}{4}))d\xi\}dz$.
Using (81), we obtain: $L_1 = O(\tau s^{-1/2}) + \hat{q}_0(\hat{T}, \hat{a}, s-\log(1-\tau)) + \alpha\hat{q}_1(\hat{T}, \hat{a}, s-\log(1-\tau)) + O(\alpha^2 s^{-2}\log s)$. By (82) and (83), we have:
$L_1 = -\frac{5\kappa}{8ps^2} + O(\tau s^{-1/2}) + O(s^{-3}) + O(\alpha^2 s^{-2}\log s)$.
$|L_2| \le C\int |\frac{z\sqrt{1-\tau}-\alpha}{\sqrt{s}} - \frac{z}{\sqrt{s-\log(1-\tau)}}|(|\hat{q}_b(\hat{T}, \hat{a}, z, s-\log(1-\tau))| + |\hat{q}_e(\hat{T}, \hat{a}, z, s-\log(1-\tau))|)\exp(-Cz^2)dz$.
Using (81) for q_b, (80) for q_e, and the fact that $q_e \equiv 0$ for $|z| \le K_0\sqrt{s}$ yields:
$L_2 \le C\{|\tau|s^{-1/2} + |\alpha|s^{-1/2}\}(s^{-2}\log s + e^{-s})$. In conclusion,
$I_1 = -\frac{5\kappa}{8ps^2} + O(\tau s^{-1/2}) + O(s^{-5/2}) + O(\alpha^2 s^{-2}\log s)$. Adding I_1, I_2 and I_3 yields (74).

We compute $\frac{\partial\hat{q}_0}{\partial\tau}$ instead of $\frac{\partial\hat{q}_0}{\partial T}$, and then we use $\frac{\partial\hat{q}_0}{\partial T} = e^s\frac{\partial\hat{q}_0}{\partial\tau}$ to conclude. With the previous notations, we write:
$\frac{\partial\hat{q}_0}{\partial\tau}(T, a, s) = \frac{\partial I_1}{\partial\tau} + \frac{\partial I_2}{\partial\tau} + \frac{\partial I_3}{\partial\tau}$.

$\frac{\partial I_3}{\partial\tau}$:
$\frac{\partial I_3}{\partial\tau} = \frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}}\frac{\kappa}{2p(s-\log(1-\tau))}(1 - \frac{1}{s-\log(1-\tau)})$, and $|\frac{\partial I_3}{\partial\tau}| \le Cs^{-1}$.

$\frac{\partial I_2}{\partial\tau}$:
$\frac{\partial I_2}{\partial\tau}$
$= \frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-\frac{1}{p-1}} + (1-\tau)^{-\frac{1}{p-1}}$
$\int d\mu(y)\chi(y,s)\frac{1}{p-1}\frac{(p-1)^2(y+\alpha)^2(1-(s-\log(1-\tau)))}{4p(1-\tau)^2(s-\log(1-\tau))^2}(p-1+\frac{(p-1)^2(y+\alpha)^2}{4p(1-\tau)(s-\log(1-\tau))})^{-1-\frac{1}{p-1}}$.
Computing as for I_2, we obtain: $\frac{\partial I_2}{\partial\tau} = O(\tau) + \frac{\kappa}{p-1} + O(s^{-1})$.

$\frac{\partial I_1}{\partial\tau}$:
$\frac{\partial I_1}{\partial\tau} = M_1 + M_2 + M_3$ with
$M_1 = \frac{1}{p-1}(1-\tau)^{-1-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)\hat{q}(\hat{T}, \hat{a}, \frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
$M_2 = \frac{1}{p-1}(1-\tau)^{-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)\frac{y+\alpha}{2(1-\tau)^{3/2}}\frac{\partial\hat{q}}{\partial y}(\hat{T}, \hat{a}, \frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
$M_3 = \frac{1}{p-1}(1-\tau)^{-\frac{1}{p-1}}\int d\mu(y)\chi(y,s)\frac{1}{1-\tau}\frac{\partial\hat{q}}{\partial s}(\hat{T}, \hat{a}, \frac{y+\alpha}{\sqrt{1-\tau}}, s-\log(1-\tau))$,
From (80), (84), and integration by parts we derive: $|\frac{\partial I_1}{\partial\tau}| \le |M_1|+|M_2|+|M_3| \le Cs^{-1/2}$.
this concludes the proof of lemma B.2.

Step 2: Behavior of $(\hat{q}_0, \hat{q}_1)$ near blow-up
We use the explicit asymptotic development given in lemma B.2 to construct a 1-manifold $\tilde{\Gamma}$ that is mapped by $(\hat{q}_0, \hat{q}_1)$ into $\partial\hat{V}_A(s)$.

Lemma B.4 (Behavior of $(\hat{q}_0,\hat{q}_1)$) *$\exists C_0=C_0(p)$, $\exists A_9>0$ $\forall A\geq A_9$, $\exists s_9(A)>0$ $\forall s\geq s_9(A)$, $\exists\Gamma_{A,s}$ rectangle in $D_{A,s}=(\hat{T},\hat{a})+(-C_0Ae^{-s}s^{-2},C_0Ae^{-s}s^{-2})\times(-C_0Ae^{-\frac{s}{2}}s^{-1},C_0Ae^{-\frac{s}{2}}s^{-1})$ such that $\forall(T,a)\in\Gamma_{A,s}$, $(\hat{q}_0,\hat{q}_1)(T,a,s)\in\partial\hat{V}_A(s)$, and $d(\Gamma_{A,s},(\hat{q}_0,\hat{q}_1)(.,.,s),0)=-1$.*

Proof:
Since $(\tilde{q}_0,\tilde{q}_1)$ given in (79) is almost the linear part of $(\hat{q}_0,\hat{q}_1)$ (see lemma B.2), we can first show for $(\tilde{q}_0,\tilde{q}_1)$ an analogous version of lemma B.4, then use lemma B.2 to conclude. We use scaling arguments to get uniform estimates in s. Indeed, let us introduce:

$$\begin{array}{rrcl}(85) & \tilde{Q}=(\tilde{Q}_0,\tilde{Q}_1):(-C_0A,C_0A)^2 & \longrightarrow & \mathbb{R}^2\\ & (\tilde{\tau},\tilde{\alpha}) & \longrightarrow & \frac{1}{A}(-\frac{5\kappa}{8p}+\tilde{\tau}\frac{\kappa}{p-1},-\tilde{\alpha}\frac{\kappa}{4p}),\end{array}$$

and

$$\begin{array}{rrcl} & \hat{Q}_s=(\hat{Q}_0,\hat{Q}_1)_s:(-C_0A,C_0A)^2 & \longrightarrow & \mathbb{R}^2\\ (86) & (\tilde{\tau},\tilde{\alpha}) & \longrightarrow & \frac{s^2}{A}(\hat{q}_0,\hat{q}_1)(\hat{T}+\frac{\tilde{\tau}}{e^s s^2},\hat{a}+\frac{\tilde{\alpha}}{e^{\frac{s}{2}}s^1},s),\end{array}$$

where $C_0=C_0(p)$. Note that $\tilde{Q}$ is independent of s, and that

$$\begin{aligned}(\tilde{q}_0,\tilde{q}_1)(T,a,s) &= \frac{A}{s^2}(\tilde{Q}_0,\tilde{Q}_1)((T-\hat{T})e^s s^2,(a-\hat{a})e^{\frac{s}{2}}s).\\ (\hat{q}_0,\hat{q}_1)(T,a,s) &= \frac{A}{s^2}(\hat{Q}_0,\hat{Q}_1)_s((T-\hat{T})e^s s^2,(a-\hat{a})e^{\frac{s}{2}}s).\end{aligned}$$

The conclusion of lemma B.4 follows if we show that there exists a 1-manifold $\tilde{\Gamma}$ in $(-C_0A,C_0A)^2$ such that $\forall(\tilde{\tau},\tilde{\alpha})\in\tilde{\Gamma}$, $\hat{Q}_s(\tilde{\tau},\tilde{\alpha})\in\partial\mathcal{C}$, and $d(\tilde{\Gamma},\hat{Q}_s,0)=-1$.
From lemma B.2, we compute for $s\geq s_{17}(A)$: $\|\tilde{Q}-\hat{Q}_s\|_{C^1((-CA,CA)^2)}\leq\frac{C\log s}{A\sqrt{s}}\to 0$ when $s\to+\infty$.
It is easy to see that $\forall\eta\in[0,1)$, $\exists\tilde{\Gamma}_\eta$ rectangle such that $\forall(\tilde{\tau},\tilde{\alpha})\in\tilde{\Gamma}_\eta$, $\tilde{Q}(\tilde{\tau},\tilde{\alpha})\in(1+\eta)\partial\mathcal{C}$, and $d(\tilde{\Gamma}_\eta,\tilde{Q},0)=-1$.
From the continuity of topological degree, we know that there exist $\eta_0>0$, $\epsilon_0>0$ such that for each curve $\tilde{\Gamma}$ (indexed by $\partial\mathcal{C}$) satisfying $\|\tilde{\Gamma}-\tilde{\Gamma}_0\|_{L^\infty(\partial C)}\leq\eta_0\sqrt{2}$ ($\tilde{\Gamma}_0$ itself is indexed by $\partial\mathcal{C}$), for each continuous function $Q:(-C_0A,C_0A)^2\longrightarrow\mathbb{R}^2$ satisfying $\|\tilde{Q}-Q\|_{L^\infty((-C_0A,C_0A)^2)}\leq\epsilon_0$, we have: $d(\tilde{\Gamma},Q,0)=-1$.
Since we have $\|\tilde{Q}-\hat{Q}_s\|_{L^\infty((-C_0A,C_0A)^2)}\leq\frac{C\log s}{A\sqrt{s}}$, and from (85) Jac $\tilde{Q}=-\frac{\kappa^2}{4p(p-1)A^2}<0$, we can take s large enough, ($s\geq s_{11}(A,\epsilon_0,\eta_0)$) so that:

$$(87)\qquad -\forall(\tilde{\tau},\tilde{\alpha})\in\tilde{\Gamma}_{\eta_0},\hat{Q}_s(\tilde{\tau},\tilde{\alpha})\in ext(1+\frac{\eta_0}{2})\mathcal{C},$$

$$(88)\qquad -\forall(\tilde{\tau},\tilde{\alpha})\in(-C_0A,C_0A)^2, Jac\hat{Q}_s(\tilde{\tau},\tilde{\alpha})<0,$$

-$\forall\omega\in Im\hat{Q}_s\cap Im\tilde{Q}$, if $\omega=\hat{Q}_s(\xi)$ then

$$(89)\qquad |\xi-\tilde{Q}^{-1}(\omega)|\leq\eta_0,$$

$$(90)\qquad -\|\tilde{Q}-\hat{Q}_s\|_{L^\infty((-C_0A,C_0A)^2)}\leq\epsilon_0.$$

By (90) and (87), we have $d(\tilde{\Gamma}_{\eta_0}, \hat{Q}_s, 0) = -1$. Therefore, by (87), $\forall \omega \in (1+\frac{\eta_0}{4})\mathcal{C}$, $d(\tilde{\Gamma}_{\eta_0}, \hat{Q}_s, \omega) = -1$ (the degree is the same in the same component of $\mathbb{R}^2 \backslash \hat{Q}_s(\tilde{\Gamma}_{\eta_0})$). Combining this with (88) and the definition of topological degree for $\mathcal{C}^1$ functions yields $\forall \omega \in (1+\frac{\eta_0}{4})\mathcal{C}$, there exists a unique$(\tilde{\tau}, \tilde{\alpha}) \in \mathbb{R}^2$ such that $\hat{Q}_s(\tilde{\tau}, \tilde{\alpha}) = \omega$. Hence, $\hat{Q}_s$ is a diffeomorphism from $(\hat{Q}_s)^{-1}((1+\frac{\eta_0}{4})\mathcal{C})$ onto $(1+\frac{\eta_0}{4})\mathcal{C}$. Thus there exists a piecewise $\mathcal{C}^1$ 1-manifold $\tilde{\Gamma}$ interior to $\tilde{\Gamma}_{\eta_0}$, such that $\hat{Q}_s$ maps $\tilde{\Gamma}$ onto ∂C ($\tilde{\Gamma}$ is diffeomorphic to $\partial\mathcal{C}$). By (89), $|\tilde{\Gamma} - \tilde{\Gamma}_{0,A}| \le \eta_0$. Therefore, we derive: $d(\tilde{\Gamma}, \hat{Q}_s, 0) = -1$. This concludes the proof of lemma B.4.

Step 3: Conclusion of the proof of lemma 4.3

We take $A \ge A_9$, $s_0 \ge \max(\hat{s}_0 + 1, s_7, s_9(A))$ and $\epsilon > 0$. $\forall s_1 > s_0$, we consider D_{A,s_1} and Γ_{A,s_1} given by lemma B.4. If $s_1 \ge s_{12}(A, \epsilon, s_0)$, then $\forall (T, a) \in \Gamma_{A,s_1}$, $|(T, a) - (\hat{T}, \hat{a})| \le \epsilon$, and $(T, a) \in D_1(s_0)$ (with the notations of lemma 4.1). Therefore, for such s_1, we have $|(T - \hat{T})e^{s_1}| \le \frac{CA}{s_1^2}$ and $|(a - \hat{a})e^{\frac{s_1}{2}}| \le \frac{CA}{s_1}$. This implies $\forall s \in [s_0, s_1]$, $|(T - \hat{T})e^{s}| \le \frac{CA}{s^2}$ and $|(a - \hat{a})e^{\frac{s}{2}}| \le \frac{CA}{s}$.
What we want to do now is to show that $\forall s \in [s_0, s_1]$, $\hat{q}(T, a, s) \in V_A(s)$. By lemma B.2, we have:
For $s_0 \ge s_{13}(A)$, $\forall (T, a) \in \Gamma_{A,s_1}$, $\forall s \in [s_0, s_1]$:

$$|\hat{q}_0(T, a, s)| \le \frac{CA}{s^2} \tag{91}$$

$$|\hat{q}_1(T, a, s)| \le \frac{CA}{s^2} \tag{92}$$

$$|\hat{q}_2(T, a, s)| \le C\frac{\log s}{s^2} \tag{93}$$

$$|\hat{q}_-(T, a, y, s)| \le C(1 + |y|^3)\frac{1}{s^2} \tag{94}$$

$$|\hat{q}_e(T, a, y, s)| \le \frac{C}{\sqrt{s}}. \tag{95}$$

Therefore, if $A \ge A_{14}$, $|\hat{q}_2(T, a, s)| \le A^2 \frac{\log s}{s^2}$,

$$|\hat{q}_-(T, a, y, s)| \le A(1 + |y|^3)\frac{1}{s^2}, |\hat{q}_e(T, a, y, s)| \le \frac{A^2}{\sqrt{s}}. \tag{96}$$

It remains for us to show that $|\hat{q}_m(T, a, s)| \le \frac{A}{s^2}$, for $m = 0, 1$.
Following the proof of lemma 3.2, we easily prove:

Lemma B.5 (Transversality property) $\exists A_{15} > 0$, $\forall A \ge A_{15}$, $\exists s_{15}(A)$ *such that* $\forall s_0 \ge s_{15}(A)$, $\forall s_1 > s_0$, *for any solution* q *of (15), satisfying:*
-Properties (91) to (95), for $s \in [s_0, s_1]$,
-$\exists s \in (s_0, s_1]$ *such that* $(q_0, q_1)(s) \in \partial\hat{V}_A(s)$,
we have the following property:
$\exists \delta > 0$ *such that* $\forall s_- \in (s - \delta, s)$, $(q_0, q_1)(s_-) \in int(\hat{V}_A(s_-))$.

If $A \ge A_{15}$ and $s_0 \ge s_{15}(A)$, then by lemma B.5, $\forall (T, a) \in \Gamma_{A,s_1}$

$$\forall s \in [s_0, s_1), (\hat{q}_0, \hat{q}_1)(T, a, s) \in int(\hat{V}_A(s)). \tag{97}$$

Indeed, this follows if we apply lemma B.5 to s_1 ($(\hat{q}_0, \hat{q}_1)(s_1) \in \partial\hat{V}_A(s_1)$ by lemma B.4) and to $s \in (s_0, s_1]$, and use
$I = \{s \in [s_0, s_1) | \forall s' \in [s, s_1), (\hat{q}_0, \hat{q}_1)(T, a, s') \in int(\hat{V}_A(s'))\}$.
The conclusion of lemma 4.3 follows for $A \geq A_6 = \max(A_9, A_{14}, A_{15})$,
$s_0 \geq \max(\hat{s}_0 + 1, s_7, s_9(A), s_{13}(A), s_{15}(A)), D_6(s_0) = D_1(s_0)$, and for $\epsilon > 0$, $s_1 = s_{12}(A, \epsilon, s_0)$ and $\Gamma_\epsilon = \Gamma_{A,s_1}$.

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Chapitre 3

Blow-up results for vector-valued nonlinear heat equations with no gradient structure

Blow-up results for vector-valued nonlinear heat equations with no gradient structure†

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1 Introduction

We are interested in the following reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u + (1+i\delta)|u|^{p-1}u, u(0,x) = u_0(x), \tag{1}$$

where, $\delta \in \mathbb{R}$, $p \in (1,+\infty)$, $p < (N+2)/(N-2)$ if $N \geq 3$, and $u_0 \in H = W^{1,p+1}(\mathbb{R}^N,\mathbb{C}) \cap L^\infty(\mathbb{R}^N,\mathbb{C})$.

(1) is a special case of the vector-valued equation:

$$\frac{\partial u}{\partial t} = \Delta u + F(u), u(x,0) = u_0(x), \tag{2}$$

where $u(t) : x \in \mathbb{R}^N \to \mathbb{R}^M$, $F : \mathbb{R}^M \to \mathbb{R}^M$ is regular and F is not necessarily a gradient.

For simplicity, we focus on the study of (1) (results for equation (2) will also be presented in section 5).

Equation (1) appears in the study of various physical problems (plasma physics, nonlinear optics). See for example Levermore and Oliver [15] and the references inside. Blow-up results for vector-valued equations have been intensively studied in differential geometry. See for example a review paper by Hamilton [12].

The Cauchy problem for equation (1) can be solved in H. $u(t)$, solution of (1) would exist either on $[0,+\infty)$ (global existence), or only on $[0,T)$, with $0 < T < +\infty$. In this case, $|u(t)|_H \to +\infty$ when $t \to T$, we say: $u(t)$ blows-up in finite time T in H. In this paper, we are interested in the finite time blow-up for equation (1).

If $\delta = 0$ and $u_0(x) \in \mathbb{R}$, then (1) can be considered as real-valued. Blow-up in this real case has been studied by various authors. Relying on the use of monotony properties and maximum principle, Ball [1] and Levine [16] find in this case obstructions to the global in time existence for (1). Other authors investigated the asymptotic behavior at blow-up of blow-up solutions of (1), $\delta = 0$. See for example Weissler [20], see for a study in the scale of *similarity variables* Giga and Kohn [11], [10], [9], Filippas and Kohn [5], Filippas and Merle [6],... The notion of asymptotic profile (that is a function from which, after a time dependent scaling, $u(t)$ approaches as $t \to T$) appears also in various papers: see for example Bricmont and Kupiainen [4], [3], Berger and Kohn [2] for a numerical study. In the scalar case and in one dimension, Herrero and Velazquez give a classification of possible blow-up profiles. They use the

† Article à paraître dans Ann. Inst. H. Poincaré Anal. Non Linéaire 15, 1998.

maximum principle and the decay in time of the number of oscillations of the solution. Some of their results are generalized to N dimensions in [19].

Most of the techniques used for $\delta = 0$ in the cited papers can not be applied in the case $\delta \neq 0$, since (1) is complex-valued (no maximum principle applied), and the equation does not derive from a gradient.

Another method has been introduced in [18] in the case $\delta = 0$ (see also [4]): Once an asymptotic profile is derived formally for (1), the existence of a solution $u(t)$ which blows-up in finite time with the suggested profile is proved rigorously, using a nonlinear analysis of equation (2) near the given profile. This approach which does not use maximum principle allows us to find blow-up solutions for vector-valued heat equations (even with no gradient structure). In this paper, we aim at adapting this method to show the existence of a blow-up solution for equation (1) with $\delta \neq 0$.

Let us remark that the scalar case provides us with a blow-up solution if $\delta = 0$. Unfortunately, this result is a one dimensional result and it fails when we perturb slightly the nonlinearity. Indeed, let us mention the case of the following vectorial equation:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u + i|u|^{q-1}u, u_{|\partial\Omega} = 0 \tag{3}$$

with $1 < q < (p+1)/2$, the method of Ball [1] yields a blow-up solution $u(t)$: $\Omega \to \mathbb{C}$ where Ω is a bounded domain of $\mathbb{R}^N$, see appendix A for details.

We show that there exists $\delta_0 > 0$ such that for each $\delta \in [-\delta_0, \delta_0]$, equation (1) has a blow-up solution. We give in addition a precise description of its blow-up behavior. Indeed,

Theorem 1 (Existence of a blow-up solution for equation (1) for small δ)
There exists $\delta_0 > 0$ such that for each $\delta \in [-\delta_0, \delta_0]$, there exist initial data u_0 such that equation (1) has a blow-up solution.

This Theorem follows directly from the following proposition which specifies the behavior of $u(t)$ near blow-up. Indeed, up to a time dependent scaling, $u(t)$ approaches a universal profile

$$(p - 1 + \frac{(p-1)^2}{4(p-\delta^2)}|z|^2)^{-\frac{1+i\delta}{p-1}} \tag{4}$$

when $t \to T$. More precisely:

Proposition 1 (Existence of a blow-up solution for equation (1) with the profile (4))
There exist $\delta_0 > 0$, $T_0 > 0$ such that for each $\delta \in [-\delta_0, \delta_0]$, for each $T \in (0, T_0]$, for each $a \in \mathbb{R}^N$,

i) there exist initial data u_0 such that equation (1) has a blow-up solution $u(x,t)$ on $\mathbb{R}^N \times [0,T)$ which blows-up in finite time T at only one blow-up point: a,

ii) moreover, we have

$$\lim_{t\to T} \|(T-t)^{\frac{1+i\delta}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f_\delta(z)^{1+i\delta}\|_{L^\infty(\mathbb{R}^N)} = 0 \tag{5}$$

$$\text{(6)} \qquad \textit{with } f_\delta(z) = (p - 1 + \frac{(p-1)^2}{4(p-\delta^2)}|z|^2)^{-\frac{1}{p-1}}.$$

iii) There exists $u_* \in \mathcal{C}(\mathbb{R}^N \backslash \{a\}, \mathbb{C})$ *such that* $u(x,t) \to u_*(x)$ *as* $t \to T$ *uniformly on compact subsets of* $\mathbb{R}^N \backslash \{a\}$*, and*

$$\text{(7)} \qquad u_*(x) \sim \left[\frac{8(p-\delta^2)|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1+i\delta}{p-1}} \quad \textit{as } x \to a.$$

Remark: Estimate (5) is really uniform in $z \in \mathbb{R}^N$. In previous papers dealing with the case $\delta = 0$, only Bricmont and Kupiainen [4] and Merle and Zaag [18] give such a uniform convergence. In most papers, the same kind convergence is proved, but only uniformly on smaller subsets (for $|z| \leq C/\sqrt{|\log(T-t)|}$ in [5],...).

Remark: In fact, we show that property *iii*) is a consequence of *ii*). We want to point out that for the heat equation (δ=0), *iii*) was known just in dimension one using the decay in time of the number of oscillations of the solution (Cf Herrero and Velazquez [13]).

Remark: To prove Proposition 1, we linearize in a way equation (1) around $f_\delta^{1+i\delta}$, and give a nonlinear finite dimensional reduction of the problem. Then, we solve the finite dimensional problem using index theory. The proof is more difficult than in [18], because of the vectorial structure, the presence of a coupling between coordinates, and the presence of one more neutral direction. These techniques give then as in [18] a stability result with respect to the initial data of the behavior described in Proposition 1 (see section 5).

Remark: Center manifold theory do not apply here. It fails to give a uniform estimate such as *ii*). One can point out that even if it works, a center manifold theory gives a convergence only uniform in the region $\{|z|\sqrt{|\log(T-t)|} \leq C\}$. For discussion in the case $\delta = 0$, see Filippas and Kohn [5], page 834-835.

Remark: We see from (6) that $0 < \delta_0 < \sqrt{p}$. Since equation (1) is rotation invariant, for each $\omega \in S^1$, we can find initial data u_0 such that the corresponding solution has the profile $f_\delta^{1+i\delta}\omega$.

From this result, one can ask: what happens for $\delta > \delta_0$? Does equation (1) still have blow-up solutions? We conjecture the existence of $\hat{\delta}_0 > 0$ such that for $|\delta| < \hat{\delta}_0$, equation (1) has blow-up solutions, while for $|\delta| > \delta_0$, no blow-up is possible for solutions of equation (1). That is, all solutions are globally defined. Indeed, from the formal asymptotic analysis, one can remark that for $|\delta| > \sqrt{p}$, $f_\delta^{1+i\delta}$ is no longer bounded, and the analysis fails. Another question arises: what happens with the critical value $\delta = \hat{\delta}_0$? Unfortunately, we are not able here to give a precise value of $\hat{\delta}_0$ and a rigorous proof of what is conjectured.

As an extension of Theorem 1, one can mention that using the same techniques, we have the same result for the following vector-valued equation:

$$\text{(8)} \qquad \frac{du}{dt} = \Delta u + |u|^{p-1}u + G(u), u(x,0) = u_0(x)$$

where

1) $u(t) : x \in \mathbb{R}^N \to \mathbb{R}^M$, $p \in (1, +\infty)$, $p < (N+2)/(N-2)$ if $N \geq 3$, $u_0 \in H = W^{1,p+1}(\mathbb{R}^N, \mathbb{R}^M) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^M)$,

2) $G : \mathbb{R}^M \to \mathbb{R}^M$ is a perturbation of $|u|^{p-1}u$ satisfying: $G(u) = G_1(|u|^2)u$, $|G(u)| \leq C|u|^r$, $|G(\lambda u_1) - G(\lambda u_2)| \leq C\lambda^r |u_1 - u_2|$ for $|u_1|, |u_2| \leq 1$, $\lambda \geq 1$, $r \in [1, p)$, $G_1 : \mathbb{R}^+ \to \mathbb{R}^+$,

Indeed,

Theorem 2 (Existence of a blow-up solution for equation (8)) .
There exist initial data u_0 such that equation (8) has a blow-up solution.

Let us mention briefly the organization of the paper. The proof of Proposition 1 relies strongly on a double-scale description of $u(t)$, solution of (1). We first give in section 2 an equivalent formulation of the problem in the scale of the well known *similarity variables* (see Giga and Kohn [11],..). Then, working in the original scale, we prove in section 3 the existence of a single-point blow-up solution for equation (1) such that (5) holds. In section 4, we return to the original scale $u(x,t)$ and use the invariance of equation (1) under the transformation $(t_0, \lambda) \to u_\lambda(x,t) = \lambda^{\frac{1+i\delta}{p-1}} u(\sqrt{\lambda}x, t_0 + \lambda t)$ to show that estimate (5) yields the equivalent (7) for the profile u_* in the original scale. We conclude in section 5 by giving some comments about the stability of the result of Proposition 1 and detailing the case of equation (8) ($M \geq 3$).

Without loss of generality, we can now assume that $a = 0$ and $N = 1$. The same proof holds in higher dimensions (see [18] for the analysis of the case $N \geq 2$). We write each complex quantity (number or function) z as $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$.

The author wants to thank Professor F. Merle for his helpful suggestions and remarks.

2 Formulation of the problem

As we mentioned just before, the proof of Proposition 1 will be completed in two steps. In the first step (section 3), it is enough to construct $u(t)$ a solution of equation (1) satisfying (5), since this implies directly that $u(t)$ blows-up in finite time T at only one blow-up point: 0 (parts *i*) and *ii*) of Proposition 1). Indeed, it easily follows from (5) that $\lim_{t\to T} |u(0,t)| = +\infty$, which means that $u(t)$ blows-up in time T at the point 0, and
$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}|u(b,t)| = 0$ for $b \neq 0$, which implies in turn that $u(t)$ does not blow-up at $b \neq 0$, and therefore blows-up only at the point 0. This last result follows directly from a Theorem by Giga and Kohn (Theorem 2.1 in [11]).

In a second step (section 4), we show how the behavior of the limiting profile $u_*(x)$ near the blow-up point (part *iii*) of Proposition 1) can be derived from the behavior of $u(t)$ as $t \to T$ given by (5).

Hence, our first goal is to construct $u(t)$ a solution of (1) satisfying (5).

To have an idea about the blow-up growth of u, solution of equation (1), we compare this solution with a blow-up solution of the corresponding differential equation
$$\frac{du}{dt} = (1 + i\delta)|u|^{p-1}u.$$

This solution is $u(t) = e^{i\theta}((p-1)(T-t))^{-\frac{1+i\delta}{p-1}}$, with $T > 0$, $\theta \in \mathbb{R}$.

Now, we consider u, a solution of equation (1) which blows-up in finite time $T > 0$ at one blow-up point $0 \in \mathbb{R}$. We expect u to grow with a similar rate near blow-up. If we introduce convenient *"similarity variables"*

$$
\begin{aligned}
y &= \frac{x}{\sqrt{T-t}} \\
s &= -\log(T-t) \\
w(y,s) &= (T-t)^{\frac{1+i\delta}{p-1}} u(x,t),
\end{aligned}
\tag{9}
$$

then, we can look for bounded non zero solutions of the following equation (which follows from (1) through (9)):

$$
\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y \nabla w - (1+i\delta)\frac{w}{p-1} + (1+i\delta)|w|^{p-1}w. \tag{10}
$$

2.1 Formal asymptotic analysis

Since equation (10) is of heat type, one can ask whether it has self-similar solutions, or at least, approximate ones. We have the following lemma:

Lemma 2.1 (Formal asymptotic behavior of w) .

i) The only self-similar solutions $w(y,s) = v_0(\frac{y}{\sqrt{s}})$ of (10) are the constant ones: $v_0 \equiv 0$, or $v_0 \equiv \kappa e^{i\theta}$, with $\kappa = (p-1)^{-\frac{1}{p-1}}$ and $\theta \in \mathbb{R}$.

ii) If equation (10) has a solution of the form

$$
w(y,s) = \sum_{j=0}^{+\infty} \frac{1}{s^j} v_j(\frac{y}{\sqrt{s}}), \tag{11}
$$

with v_j regular and bounded, then, there exists $\theta \in \mathbb{R}$ such that

$$
v_0(z) = e^{i\theta}(p-1+\frac{(p-1)^2}{4(p-\delta^2)}z^2)^{-\frac{1+i\delta}{p-1}} = e^{i\theta} f_\delta(z)^{1+i\delta}, \tag{12}
$$

where $f_\delta(z)^{1+i\delta}$ is the suggested profile in (4).

Proof:

$i)$ The equations satisfied by such a v_0 are

$$
0 = -\frac{1}{2} z v_0'(z) - (1+i\delta)\frac{v_0}{p-1} + (1+i\delta)|v_0|^{p-1}v_0, \tag{13}
$$

and $-\frac{1}{2}zv_0'(z) = v_0''(z)$. It is easy to see that the only solutions are the constant ones, and that $-\frac{v_0}{p-1} + |v_0|^{p-1}v_0 = 0$. This yields the conclusion.

$ii)$ If we substitute the form (11) in equation (10) and set $z = \frac{y}{\sqrt{s}}$, we find (if $s \to +\infty$) that v_0 satisfies (13). Searching a non constant solution $v_0(z) = \rho(z)e^{i\theta(z)}$, with $\rho > 0$, one finds that $v_0(z) = e^{i\theta}(p-1+bz^2)^{-\frac{1+i\delta}{p-1}}$, with $b > 0$, $\theta \in \mathbb{R}$.

In fact, there is only one possible value of b. Indeed, if we substitute the expanded form (11) in equation (10) and compare elements of order $\frac{1}{s}$, we obtain $F(z) = 0$, where $F(z) = \frac{1}{2}zv_0' + v_0'' - \frac{1}{2}zv_1' - (1+i\delta)\frac{v_1}{p-1} + (1+i\delta)\{(p-$

$1)|v_0|^{p-3}v_0(v_{0,1}v_{1,1}+v_{0,2}v_{1,2})+|v_0|^{p-1}v_1\}$, and $v_j=v_{j,1}+iv_{j,2}$, $j=1,2$. According to regularization properties of equation (10), it is natural to require that v_1 is $\mathcal{C}^3$, which implies that F is $\mathcal{C}^2$. $F''(0)=0$ implies $b=\frac{(p-1)^2}{4(p-\delta^2)}$. ■

Remark: Looking for approximate solutions of (10) or for solutions of (10) in the expanded form (11) is a well known approach used in various problems such as nonlinear optics, and also nonlinear heat equations (see for instance Galaktionov, Kurdyumov and Samarskii [7] for approximate self-similar solutions in the case of global existence (in time), see also Galaktionov and Vazquez [8] where an approximate solution is shown to be an admissible blow-up profile in the case of a heat equation with $(1+u)\log^2(1+u)$ as a nonlinearity). Unfortunately, computation can not be carried out easily for the form (11) in the present case, and we are unable to show the existence of a solution for equation (10) with such a form. In fact, instead of using this linear approach, we use a nonlinear one in section 3 to show that (10) actually has a solution $w(y,s)$ which approaches (in L^∞_y) $f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta}$ as $s\to+\infty$. This approach (instead of the linear one) yields the stability of such a solution (see section 5).

2.2 Transformation of the problem

Using *similarity variables* (see (9)), we see that proving (5) is equivalent to proving that (10) has a solution satisfying

$$\lim_{s\to\infty}\|w(y,s)-f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta}\|_{L^\infty}=0, \tag{14}$$

where $f_\delta^{1+i\delta}$ is given by (4).

In order to prove this, we will not linearize equation (10) around $f_\delta^{1+i\delta}$ as it suggested by (14), because the linear operator of the linearized equation has two neutral modes which are difficult to control. We will instead use modulation theory and take advantage of the invariance of (10) under the action of S^1 ($T_{\theta_0}:w\to e^{i\theta_0}w$, for each $\theta_0\in\mathbb{R}$): in fact, we introduce $q(y,s):[-\log T,+\infty)\to\mathbb{C}$ and $\theta(s):[-\log T,+\infty)\to\mathbb{R}$ such that

$$\left\{\begin{array}{rcl} w(y,s) & = & (\varphi(y,s)+q(y,s))e^{i\theta(s)} \\ 0 & = & \int\chi(y,s)(q_2(y,s)-\delta q_1(y,s))d\mu \end{array}\right. \tag{15}$$

where

$$\varphi(y,s)=\kappa^{-i\delta}(f_\delta(\frac{y}{\sqrt{s}})+\frac{\kappa}{2(p-\delta^2)s})^{1+i\delta},\ \kappa=(p-1)^{-\frac{1}{p-1}} \tag{16}$$

$$\chi(y,s)=\chi_0(\frac{|\,y\,|}{K_0s^{\frac12}}), \tag{17}$$

$\chi_0\in C_0^\infty([0,+\infty),[0,1])$, with $\chi_0\equiv1$ on $[0,1]$ and $\chi_0\equiv0$ on $[2,+\infty]$, K_0 is a constant large enough, and

$$d\mu(y)=\frac{e^{-y^2/4}}{\sqrt{4\pi}}. \tag{18}$$

The introduced liberty degree $\theta(s)$ is fixed by the second equation of (15). It will appear in the course of the proof that this second equation makes one of the neutral modes of the perturbation q to be zero, which simplifies greatly the control of q.

One can remark that we don't linearize (10) around $e^{i\theta(s)}f_\delta^{1+i\delta}$, but around $e^{i\theta(s)}\varphi$. Up to the natural action of S^1 (multiplication by $\kappa^{-i\delta}$) which simplifies the study of the linear operator of the equation on q, these two expressions differ from each other by a term of order $\frac{1}{s}$, so that (at least) some components of q are smaller that $\frac{1}{s}$, which helps to have $q(s) \to 0$ in L_y^∞ as $s \to +\infty$.

Now, we claim that proving parts i) and ii) of Proposition 1 reduces to proving the following proposition:

Proposition 2.1 (Equivalent formulation of Proposition 1, i) and ii)) *There exist* $\delta_0 > 0$, $S_0 > 0$, *such that* $\forall \delta \in [-\delta_0, \delta_0]$, $\forall s_0 \geq S_0$, $\exists q_{s_0} \in -\varphi(., s_0)+H$ *such that the system*

$$(19) \qquad \left\{ \begin{array}{rcl} \frac{\partial q}{\partial s}(y,s) & = & \{\mathcal{L}_\varphi - i\frac{d\theta}{ds}\}(q)(y,s) + B(q)(y,s) + R(\theta,y,s) \\ 0 & = & \int \chi(y,s)(q_2(y,s) - \delta q_1(y,s))d\mu(y) \end{array} \right.$$

where

$$(20) \left\{ \begin{array}{rl} \mathcal{L}_\varphi(q) & = \Delta q - \frac{1}{2}y.\nabla q - (1+i\delta)\frac{q}{p-1} \\ & +(1+i\delta)\{(p-1)|\varphi|^{p-3}\varphi(\varphi_1 q_1 + \varphi_2 q_2) + |\varphi|^{p-1}q\}, \\ B(q) & = (1+i\delta)\{|\varphi+q|^{p-1}(\varphi+q) - |\varphi|^{p-1}\varphi \\ & -(p-1)|\varphi|^{p-3}\varphi(\varphi_1 q_1 + \varphi_2 q_2) - |\varphi|^{p-1}q\}, \\ R(\theta,y,s) & = R^*(y,s) - i\frac{d\theta}{ds}\varphi, \\ R^*(y,s) & = -\frac{\partial \varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - (1+i\delta)\frac{\varphi}{p-1} + (1+i\delta)|\varphi|^{p-1}\varphi, \end{array} \right.$$

with initial data $(q(y,s_0),\theta(s_0)) = (q_{s_0}(y), 0)$ *at* $s = s_0$, *has a unique solution* (q,θ) *for* $s \geq s_0$, *satisfying* $\lim_{s\to+\infty} \|q(s)\|_{L^\infty} = 0$, *and* $\exists \theta_\infty \in \mathbb{R}$ *such that* $\theta(s) \to \theta_\infty$ *as* $s \to +\infty$.

Indeed, due to (15), the first equation in system (19) is equivalent to (10), hence, it is equivalent to (1) (use (9)). In addition, once proposition 2.1 is proved, we have: $\|w(y,s) - e^{i(\theta_\infty - \delta\log\kappa)}f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta}\|_{L^\infty}$
$\leq \|e^{i\theta(s)}(q(y,s)+\varphi(y,s)) - e^{i(\theta_\infty-\delta\log\kappa)}f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta}\|_{L^\infty}$ (use (15))
$\leq \|q(s)\|_{L^\infty} + \|(e^{i\theta(s)} - e^{i\theta_\infty})\varphi(y,s)\|_{L^\infty} + \|e^{i\theta_\infty}(\varphi(y,s) - \kappa^{-i\delta}f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta})\|_{L^\infty}$
$\leq \|q(s)\|_{L^\infty} + C|\theta(s)-\theta_\infty| + Cs^{-1} \to 0$ as $s \to +\infty$ (see (16)).

Therefore, $w(y,s)$ approaches $e^{i(\theta_\infty-\delta\log\kappa)}f_\delta(\frac{y}{\sqrt{s}})^{1+i\delta}$ in $L^\infty(\mathbb{R})$ as $s \to +\infty$. Since (10) is rotation invariant, we can replace w by $e^{-i(\theta_\infty-\delta\log\kappa)}w$ to obtain (14), which is equivalent to (5) through similarity variables (see (9)).

Hence, we must study system (19) for $(q,\theta) \in L^\infty(\mathbb{R}) \times \mathbb{R}$ to solve the problem. Its evolution is mostly influenced by its linear part $\mathcal{L}_{\varphi,\theta}(q) = (\mathcal{L}_\varphi - i\frac{d\theta}{ds})(q)$. Let us study more carefully this operator. $\mathcal{L}_{\varphi,\theta}$ is a $\mathbb{R}$-linear operator defined on $\mathcal{D}(\mathcal{L}_{\varphi,\theta}) \subset L^2(\mathbb{R},\mathbb{C},d\mu)$. Since we are interested in the behavior of $(q(s),\theta(s))$ in $L^\infty(\mathbb{R}) \times \mathbb{R}$ as $s \to +\infty$, let us consider the limit as $s \to +\infty$ of $\mathcal{L}_{\varphi,\theta}(r)$ for a fixed $r \in L^\infty(\mathbb{R},\mathbb{C})$ (note that $L^\infty(\mathbb{R},\mathbb{C}) \subset L^2(\mathbb{R},\mathbb{C},d\mu)$).

Since $\theta(s)$ will be shown to have a limit when $s \to +\infty$, we can think that the effect of $\frac{d\theta}{ds}$ appearing in the expression of $\mathcal{L}_{\varphi,\theta}$ (see (20)) will be negligible. Therefore, $\mathcal{L}_{\varphi,\theta}(r) \to \tilde{\mathcal{L}}(r) = \Delta r - \frac{1}{2}y.\nabla r + (1+i\delta)r_1$ as $s \to +\infty$ (see (20) and (16)). The following lemma provides us with the spectral decomposition of $\tilde{\mathcal{L}}$:

Lemma 2.2 (Eigenvalues of $\tilde{\mathcal{L}}$) .

i) $\tilde{\mathcal{L}}$ *is a* $\mathbb{R}-$*linear operator defined on* $L^2(\mathbb{R},\mathbb{C},d\mu)$ *and its eigenvalues are given by* $\{1-\frac{m}{2}|m\in\mathbb{N}\}$. *Its eigenfunctions are given by* $\{(1+i\delta)h_m, ih_m|m\in\mathbb{N}\}$ *where*

$$h_m(y)=\sum_{n=0}^{[\frac{m}{2}]}\frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{21}$$

We have: $\tilde{\mathcal{L}}((1+i\delta)h_m)=(1-\frac{m}{2})(1+i\delta)h_m$ *and* $\tilde{\mathcal{L}}(ih_m)=-\frac{m}{2}ih_m$.

ii) Each $r\in L^2(\mathbb{R},\mathbb{C},d\mu)$ *can be uniquely written as* $r(y)=(1+i\delta)(\sum_{m=0}^{+\infty}\hat{r}_{1,m}h_m(y))+i(\sum_{m=0}^{+\infty}\hat{r}_{2,m}h_m(y))$, *where* $\hat{r}_{j,m}\in\mathbb{R}$.

Proof:

i) From [18], we know that $\{h_m|m\in\mathbb{N}\}$ is a total family in $L^2(\mathbb{R},\mathbb{R},d\mu)$, and that $(\Delta-\frac{1}{2}y.\nabla)h_m=-\frac{m}{2}h_m$. Hence, we decompose each $r\in L^2(\mathbb{R},\mathbb{C},d\mu)$ as $r(y)=\sum_{m=0}^{+\infty}(r_{1,m}+ir_{2,m})h_m(y)$.

$\lambda\in\mathbb{R}$ is an eigenvalue for $\tilde{\mathcal{L}}\iff\exists r\in L^2(\mathbb{R},\mathbb{C},d\mu), r\neq 0, \tilde{\mathcal{L}}r=\lambda r$

$$\iff \exists r\neq 0\ \forall m\in\mathbb{N}\begin{cases}(1-\frac{m}{2}-\lambda)\quad r_{1,m} & =0\\ \lambda\quad r_{1,m}\quad +(-\frac{m}{2}-\lambda)r_{2,m} & =0\end{cases}$$

$\iff\exists m\in\mathbb{N}\ \lambda=1-\frac{m}{2}$

The computation of eigenfunctions is easy and we shall skip it.

ii) We write $r=(1+i\delta)\tilde{r}_1+i\tilde{r}_2$, with $\tilde{r}_j\in L^2(\mathbb{R},\mathbb{R},d\mu)$, and use the fact that $\{h_m|m\in\mathbb{N}\}$ is a total family in $L^2(\mathbb{R},\mathbb{R},d\mu)$. ■

Let us consider $(q(s),\theta(s))$ a solution of system (19). We will use an integral formulation of its first equation in terms of the fundamental solution of $\mathcal{L}_\varphi$. We want $\|q(s)\|_{L^\infty}\to 0$ as $s\to+\infty$. This L^∞ control will result from the L^∞ control of $(1-\chi(y,s))q(y,s)$ and $\chi(y,s)q(y,s)$ (see (17) for χ):

1) in the "regular" region $|y|\geq K_0\sqrt{s}$, $\mathcal{L}_\varphi$ behaves in $L^2(\mathbb{R},\mathbb{C},d\mu)$ like an operator with a fully negative spectrum. We will show from (20) that the fundamental solution of $\mathcal{L}_\varphi$ between s_0 and $s_1>s_0$ is a strict contraction from $L^\infty(|y|\geq K_0\sqrt{s})$ to $L^\infty(\mathbb{R})$. Therefore, the control of $(1-\chi(y,s))q(y,s)$ in $L^\infty(\mathbb{R})$ will be done without difficulties.

2) in the "singular" region $|y|\leq K_0\sqrt{s}$, $\mathcal{L}_\varphi$ behaves in $L^2(\mathbb{R},\mathbb{C},d\mu)$ like $\tilde{\mathcal{L}}$. In order to control $\chi q(y,s)$, we expand it with respect to the spectrum of $\tilde{\mathcal{L}}$ in $L^2(\mathbb{R},\mathbb{C},d\mu)$, but we will control χq in $L^\infty(\mathbb{R})$ and not only in $L^2(\mathbb{R},\mathbb{C},d\mu)$ (see section 3 for the rigorous analysis).

By lemma 2.2, $\tilde{\mathcal{L}}$ has two expanding directions $((1+i\delta)h_0,(1+i\delta)h_1)$, two null ones $((1+i\delta)h_2, ih_0)$ and countably many negative ones.

Here, the situation is a bit more complicated than in [18], because we have two null directions (instead of only one).

Our strategy to control all the components of χq so that $\|\chi q(s)\|_{L^\infty}\to 0$ as $s\to+\infty$ is to control the part of χq corresponding to the negative spectrum of $\tilde{\mathcal{L}}$ and the one parallel to $(1+i\delta)h_2$ (which corresponds to the null eigenvalue) as in [18]. The component parallel to ih_0 (which corresponds also to the null eigenvalue) has been fixed by the second equation of (19) to be zero (using modulation theory and the phase invariance of the equation).

However, the analysis of system (19) is longer than the equivalent analysis in [18], because of terms with $\frac{d\theta}{ds}$, and the presence of strong coupling between the two scalar parts: $\tilde{q}_1$ and $\tilde{q}_2$ of q, satisfying: $q = (1+i\delta)\tilde{q}_1 + i\tilde{q}_2$. Fortunately, $\frac{d\theta}{ds}$ will be controlled near the profile φ (see 16), and, although the coupling will be of critical size, its effect will be controlled by δ, which can be chosen small.

3 Existence of a blow-up solution for equation (2)

In this section, we prove proposition 2.1, which implies parts $i)$ and $ii)$ of Proposition 1 and then Theorem 1.

3.1 Geometrical property for q

As in [18], the convergence of $\|q(s)\|_{L^\infty}$ to zero as $s \to +\infty$ will follow from a geometrical property: $q(s) \in V_A(s)$, where $V_A(s) \subset L^\infty(\mathbb{R},\mathbb{C})$ shrinks to $q \equiv 0$ as $s \to +\infty$. The structure of $V_A(s)$ respects the free-boundary moving in q at the rate $\sqrt{s}$, and also the eigenfunctions of the operator $\tilde{\mathcal{L}}$ (Cf lemma 2.2).

In order to define $V_A(s)$, we introduce the following useful notations:

For each $g \in L^\infty(\mathbb{R},\mathbb{R})$ and $s > 0$, we define $g_b(y,s) = \chi(y,s)g(y)$ and $g_e(y,s) = (1-\chi(y,s))g(y)$. Since $L^\infty(\mathbb{R},\mathbb{R}) \subset L^2(\mathbb{R},\mathbb{R},d\mu)$, we introduce for each $m \in \mathbb{N}$, $g_m(s)$ as the $L^2(\mathbb{R},\mathbb{R},d\mu)$ projection of $g_b(y,s)$ on h_m, (Cf (21)). We also let $g_-(y,s) = P_-(g_b)$ and $g_\perp(y,s) = P_\perp(g_b)$, where P_- and $P_\perp$ are the $L^2(\mathbb{R},\mathbb{R},d\mu)$ projectors respectively on Vect $\{h_m | m \geq 3\}$ and Vect $\{h_m | m \geq 1\}$. Thus, we write either

$$g(y) = \sum_{m=0}^{2} g_m(s)h_m(y) + g_-(y,s) + g_e(y,s) \tag{22}$$

or

$$g(y) = g_0(s)h_0(y) + g_\perp(y,s) + g_e(y,s). \tag{23}$$

For each $z \in \mathbb{C}$, we write in a unique way $z = (1+i\delta)\tilde{z}_1 + i\tilde{z}_2$, where $\tilde{z}_1$ and $\tilde{z}_2$ are real.

Hence, if $r \in L^\infty(\mathbb{R},\mathbb{C})$, we write: $r(y) = (1+i\delta)\tilde{r}_1(y) + i\tilde{r}_2(y)$ and expand $\tilde{r}_1$ and $\tilde{r}_2$ respectively as in (22) and (23). Thus, we write: $r(y) = (1+i\delta)\tilde{r}_1(y) + i\tilde{r}_2(y)$

$$\begin{aligned} &= (1+i\delta)\{\textstyle\sum_{m=0}^{2} \tilde{r}_{1,m}(s)h_m(y) + \tilde{r}_{1,-}(y,s) + \tilde{r}_{1,e}(y,s)\} \\ &+ \quad i\{\tilde{r}_{2,0}(s)h_0(y) + \tilde{r}_{2,\perp}(y,s) + \tilde{r}_{2,e}(y,s)\}. \end{aligned} \tag{24}$$

Definition 3.1 *For each $A > 0$, for each $s > 0$, let $V_A(s)$ be the set of all functions r in $L^\infty(\mathbb{R},\mathbb{C})$ such that*

$$\begin{array}{llllll} |\tilde{r}_{1,m}(s)| & \leq & As^{-2}, & for\ m = 0,1, & & \\ |\tilde{r}_{1,2}(s)| & \leq & A^2(\log s)s^{-2}, & |\tilde{r}_{2,0}(s)| & \leq & As^{-2}, \\ |\tilde{r}_{1,-}(y,s)| & \leq & A(1+|y|^3)s^{-2}, & |\tilde{r}_{2,\perp}(y,s)| & \leq & A(1+|y|^3)s^{-2}, \\ \|\tilde{r}_{1,e}(s)\|_{L^\infty} & \leq & A^2 s^{-\frac{1}{2}}, & \|\tilde{r}_{2,e}(s)\|_{L^\infty} & \leq & A^2 s^{-\frac{1}{2}}, \end{array}$$

where r is given by (24).

Remark: We note that $L^\infty(\mathbb{R},\mathbb{C}) \subset L^2(\mathbb{R},\mathbb{C},d\mu)$, which justifies the expansion with respect to the eigenvalues of $\tilde{\mathcal{L}}$ in definition 3.1.

Remark: It is easy to see that if $q(s) \in V_A(s)$, then $\forall y \in \mathbb{R}$, $|q(y,s)| \leq C(A)s^{-1/2}$ (see [18] for details). Therefore, $\|q(s)\|_{L^\infty(\mathbb{R},\mathbb{C})} \to 0$ as $s \to +\infty$, and we obtain a convergence in $L^\infty(\mathbb{R},\mathbb{C})$ and not only in $L^2(\mathbb{R},\mathbb{C},d\mu)$, as in other papers (see [5],..). We emphasize that a convergence in $L^2(\mathbb{R},\mathbb{C},d\mu)$ or more generally in $H^m(\mathbb{R},\mathbb{C},d\mu)$ yields a convergence in $L^\infty([-R,R],\mathbb{C})$ for each $R>0$, and never a uniform convergence on $\mathbb{R}$.

With this remark, we claim that proposition 2.1 follows from the following proposition:

Proposition 3.1 Equivalent formulation of Proposition 1, $i)$ **and** $ii)$ *There exists $A>0$, $\delta_0>0$, $S_0>0$, such that $\forall\delta \in [-\delta_0,\delta_0]$, $\forall s_0 \geq S_0$, $\exists(d_0,d_1)\in\mathbb{R}^2$ such that system (19) with initial data at $s=s_0$*

$$(25)\left\{\begin{array}{rcl} q_{d_0,d_1}(y,s_0) &=& (1+i\delta)f_0(\frac{y}{\sqrt{s_0}})^p(d_0+d_1y/\sqrt{s_0}) - (\frac{\alpha}{s_0})^{1+i\delta} \\ &+& i\frac{\alpha}{s_0}(\sin[\delta\log(\frac{\alpha}{s_0})] - \delta\cos[\delta\log(\frac{\alpha}{s_0})])f_0(\frac{y}{\sqrt{s_0}})^p\beta(s_0) \\ \theta(s_0) &=& 0 \end{array}\right.$$

(where f_0 is given by (6),

$$(26)\qquad \alpha = \frac{\kappa}{2(p-\delta^2)},\ \beta(s_0) = \frac{\int f_0(\frac{y}{\sqrt{s_0}})^p\chi(y,s_0)d\mu(y)}{\int\chi(y,s_0)d\mu(y)})$$

has a unique solution $(q,\theta)_{d_0,d_1}$ for $s\geq s_0$, satisfying $q(s)\in V_A(s)$, $\forall s\geq s_0$.

Indeed, once proposition 3.1 is proved, we take for q_{s_0} the expression in (25). From $q(s)\in V_A(s)$, $\forall s\geq s_0$, we have $\|q(s)\|_{L^\infty}\to 0$ as $s\to+\infty$, and $\exists\theta_\infty$ such that $\theta(s)\to\theta_\infty$ as $s\to+\infty$. Indeed, we have the following lemma:

Lemma 3.1 *$\forall A>0$, $\exists s_3(A)>0$ such that $\forall\delta\in[-1,1]$, $\forall s\geq s_3(A)$, if $q(s)\in V_A(s)$, then $|\frac{d\theta}{ds}(s)|\leq\frac{C}{s^2}$.*

This lemma implies $\int_{s_0}^{+\infty}|\frac{d\theta}{ds}(s)|ds<+\infty$, which gives θ_∞ such that $\theta(s)\to\theta_\infty$ as $s\to+\infty$. We give the proof of this lemma in the next subsection.

In order to understand the dynamics of q and θ, we derive the equations satisfied by $\tilde{q}_1$ and $\tilde{q}_2$ ($q(y,s)=(1+i\delta)\tilde{q}_1(y,s)+i\tilde{q}_2(y,s)$, Cf decomposition (24)) and θ:

Lemma 3.2 (Equations satisfied by $\tilde{q}_1$, $\tilde{q}_2$ and θ) *If q satisfies (19) for $s\geq s_0$, then:*

$$\begin{aligned} \frac{\partial\tilde{q}_1}{\partial s}(y,s) &= (\mathcal{L}+V_{1,1}(y,s)+\delta\frac{d\theta}{ds}(s))\tilde{q}_1+(V_{1,2}(y,s)+\frac{d\theta}{ds}(s))\tilde{q}_2 \\ (27)\qquad &+ \tilde{B}_1(q(y,s))+\tilde{R}_1(\theta,y,s), \\ \frac{\partial\tilde{q}_2}{\partial s}(y,s) &= (V_{2,1}-(1+\delta^2)\frac{d\theta}{ds}(s))\tilde{q}_1+(\mathcal{L}-1+V_{2,2}(y,s)-\delta\frac{d\theta}{ds}(s))\tilde{q}_2 \\ (28)\qquad &+ \tilde{B}_2(q(y,s))+\tilde{R}_2(\theta,y,s), \end{aligned}$$

$$\frac{d\theta}{ds}\int\chi(y,s)((1+\delta^2)\tilde\varphi_1+\delta\tilde\varphi_2+(1+\delta^2)\tilde q_1+\delta\tilde q_2)d\mu$$
$$=\int\chi(\mathcal{L}-1)\tilde q_2 d\mu+\int\frac{\partial\chi}{\partial s}\tilde q_2 d\mu+\int\chi(V_{2,1}\tilde q_1+V_{2,2}\tilde q_2)d\mu$$
$$+\int\chi\tilde B_2(q)d\mu+\int\chi(y,s)\tilde R_2^*(y,s), \tag{29}$$

where

$$\mathcal{L}=\Delta-\frac{1}{2}y.\nabla+1, \tag{30}$$

$$\begin{cases} V_{1,1}(y,s) &= (1-\delta^2)(|\varphi|^{p-1}-\frac{1}{p-1})+(p-1)|\varphi|^{p-3}(\varphi_1^2-\delta^2\varphi_2^2)-1\\ V_{1,2}(y,s) &= -\delta(|\varphi|^{p-1}-\frac{1}{p-1})+(p-1)|\varphi|^{p-3}(\varphi_1-\delta\varphi_2)\varphi_2\\ V_{2,1}(y,s) &= (1+\delta^2)\{\delta(|\varphi|^{p-1}-\frac{1}{p-1})+(p-1)|\varphi|^{p-3}(\varphi_1+\delta\varphi_2)\varphi_2\}\\ V_{2,2}(y,s) &= (1+\delta^2)\{(|\varphi|^{p-1}-\frac{1}{p-1})+(p-1)|\varphi|^{p-3}\varphi_2^2\},\end{cases}$$

φ is given by (16), $(1+i\delta)\tilde B_1+i\tilde B_2=B$, $(1+i\delta)\tilde R_1+i\tilde R_2=R$, and B, R are given by (20).

Proof: (27) and (28) follow directly from (19). For (29), we note that we derive form (19) $\frac{d}{ds}\int\chi(y,s)\tilde q_2(y,s)d\mu(y)=0$ ($\tilde q_2=q_2-\delta q_1$). Therefore $\int\chi(y,s)\frac{\partial\tilde q_2}{\partial s}(y,s)d\mu(y)=-\int\frac{\partial\chi}{\partial s}(y,s)\tilde q_2(y,s)d\mu(y)$. Multiplying (28) by χ and integrating with respect to $d\mu$ yields (29). ■

The proof of Proposition 3.1 follows the general ideas developed in [18]. Indeed, it is divided in two parts:

-In a first part, we reduce the problem of the control in $V_A(s)$ of all the components of $q(s)$ to the problem of controlling $(\tilde q_{1,0}(s),\tilde q_{1,1}(s))$, which are the components of q corresponding to expanding directions of $\tilde{\mathcal{L}}$ (see (24) and lemma 2.2). That is, we reduce an infinite dimensional problem to a finite dimensional one.

-The second part of the proof is devoted to the solving of the finite dimensional problem, using 2-dimensional dynamics of $(\tilde q_{1,0},\tilde q_{1,1})(s)$ and a topological argument (index theory) based on the variation of the 2-dimensional parameter (d_0,d_1) appearing in the expression (25) of initial data $q_{d_0,d_1}(y,s_0)$.

3.2 Proof of the geometrical property on $q(s)$

First, we prove lemma 3.1 which insures that proposition 3.1 implies proposition 2.1 and then Proposition 1 *i*) and *ii*).

Proof of lemma 3.1:

We control $\frac{d\theta}{ds}$ thanks to equation (29). Let us estimate each term appearing in:

If $s_0\ge s_3(A)$, we have the following estimates.

- Since $q\in V_A$, the left-hand side of (29) is (in absolute value) greater than $C|\frac{d\theta}{ds}|$ where $C>0$.

- Since $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R},d\mu)$, $\int\chi(\mathcal{L}-1)\tilde q_2 d\mu=\int(\mathcal{L}-1)\chi\tilde q_2 d\mu=\int(\frac{\partial^2\chi}{\partial y^2}-\frac{1}{2}y\frac{\partial\chi}{\partial y})\tilde q_2 e^{-y^2/8}\frac{e^{-y^2/8}}{\sqrt{4\pi}}dy$. From (17), $|\frac{\partial^2\chi}{\partial y^2}-\frac{1}{2}y\frac{\partial\chi}{\partial y}|\le C$, and $\frac{\partial^2\chi}{\partial y^2}-\frac{1}{2}y\frac{\partial\chi}{\partial y}\equiv 0$ for $|y|\le K_0\sqrt{s}$. Hence, we can bound $e^{-y^2/8}$ by $e^{-K_0^2 s/8}$, and use $q(s)\in V_A(s)$ to obtain $|\int\chi(\mathcal{L}-1)\tilde q_2 d\mu|\le Ce^{-s}$ (if K_0 is large enough).

- The same argument yields $|\int \frac{\partial\chi}{\partial s}\tilde{q}_2 d\mu| \le Ce^{-s}$.
- We have $|V_{i,j}(y,s)| \le Cs^{-1}(1+|y|^2)$ (see lemma B.1 in appendix B). Combining this with Definition 3.1, we get $|\int \chi(V_{2,1}\tilde{q}_1 + V_{2,2}\tilde{q}_2)d\mu| \le Cs^{-3}\log s$.
- We have $|\chi(y,s)B(q(y,s))| \le C|q|^2$ for $q(s) \in V_A(s)$ (see lemma B.4). Therefore, $|\int \chi\tilde{B}_2(q)d\mu| \le \int \chi|q|^2 d\mu \le Cs^{-3}$.
- From (20), $|\int \chi(y,s)\tilde{R}_2^*(y,s)| \le \frac{C}{s^2}$ (see lemma B.5).

Combining all the previous estimates gives: $|\frac{d\theta}{ds}| \le \frac{C}{s^2}$. ■

Now, we give the proof of proposition 3.1 following the plan announced in the previous subsection.

Part I: Reduction to a finite dimensional problem

Here, (q,θ) stands for a solution of system (19) with initial data (25). We show through a priori estimates that finding $(d_0,d_1) \in \mathbb{R}^2$ such that $\forall s \ge s_0$ $q(s) \in V_A(s)$ is equivalent to finding $(d_0,d_1) \in \mathbb{R}^2$ such that $\forall s \ge s_0$ $(\tilde{q}_{1,0}(s), \tilde{q}_{1,1}(s)) \in \hat{V}_A(s)$, where

Definition 3.2 *For each $A > 0$, for each $s > 0$, we define $\hat{V}_A(s)$ as being the set $[-\frac{A}{s^2}, \frac{A}{s^2}]^2 \subset \mathbb{R}^2$.*

Proposition 3.2 (Control of $q(s)$ by $(\tilde{q}_{1,0}(s), \tilde{q}_{1,1}(s))$ in $\tilde{V}_A(s)$) *There exists $A_1 > 0$ such that for each $A \ge A_1$, there exists $\delta_1(A) > 0$, $s_1(A) > 0$ such that for each $\delta \in [-\delta_1, \delta_1]$, $s_0 \ge s_1(A)$, we have the following properties:*
-if (d_0,d_1) is chosen so that $(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1}(s_0)) \in \hat{V}_A(s_0)$, and,
-if for $s_1 \ge s_0$, we have $\forall s \in [s_0,s_1]$, $q(s) \in V_A(s)$ and $q(s_1) \in \partial V_A(s_1)$,
then

i) $(\tilde{q}_{1,0}(s_1), \tilde{q}_{1,1}(s_1)) \in \partial\hat{V}_A(s_1)$,

ii) **(transversality)** *there exists $\eta_0 > 0$ such that $\forall \eta \in (0,\eta_0)$, $(\tilde{q}_{1,0}(s_1+\eta), \tilde{q}_{1,1}(s_1+\eta)) \notin \hat{V}_A(s_1+\eta)$ (hence, $q(s_1+\eta) \notin V_A(s_1+\eta)$).*

Proof: see Proof of Proposition 3.2 below.

Now, we fix $A \ge A_1$, and $\delta_0 = \delta_1$. We note $q(d_0,d_1) = q_{d_0,d_1}$ (see proposition 3.1).

Part II: Topological argument for the finite dimensional problem

In the following proposition, we initialize the finite dimensional problem and study the Cauchy problem for system (19).

Proposition 3.3 (Initialization and Cauchy problem for system (19)) *There exists $s_2(A) > 0$ such that for each $\delta \in [-\delta_0, \delta_0]$, for each $s_0 \ge s_2(A)$,*

i) there exists a set $\mathcal{D}_{s_0} \subset \mathbb{R}^2$ topologically equivalent to a square with the following property:

$$q(d_0,d_1,s_0) \in V_A(s_0) \text{ if and only if } (d_0,d_1) \in \mathcal{D}_{s_0}.$$

ii) For each $(d_0,d_1) \in \mathcal{D}_{s_0}$, $\exists S = S(d_0,d_1) > s_0$ (maximal) such that system (19) with initial data (25) at $s = s_0$ has a unique solution $(q,\theta)(d_0,d_1)$ on $[s_0,S)$, with q and θ $\mathcal{C}^2$ and $q(s) \in V_{A+1}(s)$, $\forall s \in [s_0,S)$.

iii) (q,θ) is continuous with respect to (d_0,d_1,s).

Proof:

$i)$ From (25), we have
$\tilde{q}_1(d_0,d_1,y,s_0) = f_0(\frac{y}{\sqrt{s_0}})^p(d_0 + d_1\frac{y}{\sqrt{s_0}}) - \frac{\alpha}{s_0}\cos[\delta\log(\frac{\alpha}{s_0})]$ and
$\tilde{q}_2(d_0,d_1,y,s_0) = -\frac{\alpha}{s_0}(\delta - \sin[\delta\log(\frac{\alpha}{s_0})])(1-\beta(s_0)f_0(\frac{y}{\sqrt{s_0}})^p)$. The expression of $\tilde{q}_1$ is similar to the expression of initial data (33) for the similar equation (15) in [18]. $\tilde{q}_2$ is a sum of two terms appearing in the mentioned formula (33) in [18]. Hence, one can adapt without difficulties lemmas 3.1 and 3.3 of [18] to conclude (note that $\tilde{q}_{2,0}(d_0,d_1,s_0)=0$).

$ii)$ As if to use (15) in a reverse way, we introduce

$$w(y,s) = e^{i\theta(s)}(q(y,s)+\varphi(y,s)). \tag{31}$$

Therefore, our problem is equivalent to the following system in (w,θ):

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - (1+i\delta)\frac{w}{p-1} + (1+i\delta)|w|^{p-1}w \tag{32}$$

$$F((\theta(s),s) = 0$$

where $F(\theta,s) =$

$$\cos(\theta)(w_{2,0}(s) - \delta w_{1,0}(s)) + \sin(\theta)(-w_{1,0}(s) - \delta w_{2,0}(s)) - \tilde{\varphi}_{2,0}(s), \tag{33}$$

with initial data

$$w(d_0,d_1,s_0) = q(d_0,d_1,s_0) + \varphi(s_0), \tag{34}$$

$$\theta(s_0) = 0. \tag{35}$$

By a simple calculation, we have $w(d_0,d_1,s_0)\in H$. Hence from classical theory, we have local existence and uniqueness of a $\mathcal{C}^2$ solution for (32) with initial data (34).

In order to prove existence and uniqueness for $\theta(s)$, we apply the implicit function theorem to F near $(\theta,s) = (0,s_0)$. First we compute $\frac{\partial F}{\partial\theta}(\theta,s) = -\sin(\theta)(w_{2,0}(s) - \delta w_{1,0}(s)) + \cos(\theta)(-w_{1,0}(s) - \delta w_{2,0}(s))$ and $\frac{\partial F}{\partial\theta}(0,s_0) = -\varphi_{1,0}(s_0) - \delta\varphi_{2,0}(s_0) - (1+\delta^2)\tilde{q}_{1,0}(s_0) - \delta\tilde{q}_{2,0}(s_0)$ (use (31)). By (16), $-\varphi_{1,0}(s_0) - \delta\varphi_{2,0}(s_0) \to -\kappa$ as $s_0\to+\infty$. Hence, if $s_0\ge s_2(A)$ and $(d_0,d_1)\in\mathcal{D}_{s_0}$, then $q(s_0)\in V_A(s_0)\subset V_{A+1}(s_0)$ and $\frac{\partial F}{\partial\theta}(0,s_0)\neq 0$. Since $F(0,s_0)=0$ (because $\tilde{q}_{2,0}(d_0,d_1,s_0)=0$), and F is $\mathcal{C}^2$, we have existence and uniqueness of $\mathcal{C}^2$ $\theta(s)$.

We add that the solution $(q,\theta)(s)$ is well defined if we require $q(s)\in V_{A+1}(s)$.

$iii)$ Using again the equivalent formulation (31), we see that $(q,\theta)(d_0,d_1,s)$ is a continuous function of $(q(d_0,d_1,s_0),s)$. Since $q(d_0,d_1,s_0)$ is continuous in (d_0,d_1) (it is affine, see (25)), we obtain $iii)$. ■

Now, we fix $S_0 > \max(s_1(A),s_2(A))$, and take $\delta\in[-\delta_0,\delta_0]$, $s_0\ge S_0$. Then we start the proof of Proposition 3.1 for A, δ and s_0.

We argue by contradiction: According to proposition 3.3, for each $(d_0,d_1)\in\mathcal{D}_{s_0}$, system (19) with initial data (25) has a unique solution on $[s_0,S(d_0,d_1))$ and $q(d_0,d_1,s_0)\in V_A(s_0)$. We suppose then that for each $(d_0,d_1)\in\mathcal{D}_{s_0}$, there exists

$s > s_0$ such that $q(d_0, d_1, s) \notin V_A(s)$. Let $s_*(d_0, d_1)$ be the infimum of all these s. By proposition 3.2 ($s_1 = s_*$), we can define the following function:

$$\begin{array}{rcl} \Phi : \mathcal{D}_{s_0} & \longrightarrow & \partial\mathcal{C} \\ (d_0, d_1) & \longrightarrow & \dfrac{s_*(d_0, d_1)^2}{A}(\tilde{q}_{1,0}, \tilde{q}_{1,1})(d_0, d_1, s_*(d_0, d_1)) \end{array}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Now we claim

Proposition 3.4 *i) Φ is a continuous mapping from $\mathcal{D}_{s_0}$ to $\partial\mathcal{C}$.*
ii) There exists a non-trivial affine function $g : \mathcal{D}_{s_0} \to \mathcal{C}$ such that $\Phi \circ g^{-1}_{|\partial\mathcal{C}} = Id_{|\partial\mathcal{C}}$.

From that , a contradiction follows (Index Theory). Hence, there exists (d_0, d_1) such that $\forall s \geq s_0$, $q(d_0, d_1, s) \in V_A(s)$.
This concludes the proof of proposition 3.1, and also the proof of parts $i)$ and $ii)$ of Proposition 1. ■

Proof of proposition 3.4:
$i)$ Part $iii)$ of proposition 3.3 implies that $(\tilde{q}_{1,0}(s), \tilde{q}_{1,1}(s))$ is a continuous function of (d_0, d_1). Using the transversality property of $(\tilde{q}_{1,0}(s_*), \tilde{q}_{1,1}(s_*))$ on $\partial\hat{V}_A(s_*)$ ($ii)$ of proposition 3.2), we claim that $s_*(d_0, d_1)$ is continuous. Therefore, Φ is continuous.

$ii)$ If $(d_0, d_1) \in \partial\mathcal{D}_{s_0}$, then from $i)$ of proposition 3.3 , $q(d_0, d_1, s_0) \in V_A(s_0)$. According to the proof of lemma 3.3 in [18], $(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1}(s_0)) \in \partial\hat{V}_A(s_0)$. Applying $ii)$ of proposition 3.2 with s_0 and $s_1 = s_0$, we have $s_*(d_0, d_1) = s_0$, and $\Phi(d_0, d_1) = \frac{s_0^2}{A}(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1}(s_0))$.
Let $T : (d_0, d_1) \in \mathcal{D}_{s_0} \to \frac{s_0^2}{A}(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1})(s_0)) \in \mathcal{C}$. From (25), T is affine. Hence $\Phi \circ T^{-1}_{|\partial\mathcal{D}_{s_0}} = Id_{|\partial\mathcal{D}_{s_0}}$. This concludes the proof of proposition 3.4. ■

Now, we give the proof of proposition 3.2.

3.3 Proof of proposition 3.2

As we suggested in the formulation of the problem, the proof follows the general ideas of [18]. However, it is more complicated because of terms with $\frac{d\theta}{ds}$ or because of strong interference between $\tilde{q}_1$ and $\tilde{q}_2$ (see (27), (28)). Therefore, we summarize arguments which are similar to those exposed in [18] by showing how to adapt them to the present context, and emphasize the arguments relative to these new terms.

We divide the proof in three steps:

- In Step 1, we give a priori estimates on $q(s)$ in $V_A(s)$: assume that for given $A > 0$ large, $\rho > 0$ and an initial time $s_0 \geq s_4(A, \rho)$, we have $q(s) \in V_A(s)$ for each $s \in [\sigma, \sigma + \rho]$, where $\sigma \geq s_0$. Using system (19) which is satisfied by q, we then derive new bounds on $\tilde{q}_{1,2}$, $\tilde{q}_{1,-}$, $\tilde{q}_{1,e}$, $\tilde{q}_{2,\perp}$ and $\tilde{q}_{2,e}$ in $[\sigma, \sigma + \rho]$ (involving A and ρ).

-In Step 2, we show that these new bounds are better than those defining $V_A(s)$ (see definition 3.1) provided that $\rho \leq \rho^*(A)$. Since $\tilde{q}_{1,2}(s) = 0$ by hypothesis in (19), only $\tilde{q}_{1,0}(s)$ and $\tilde{q}_{1,1}(s)$ remain to be controlled: the problem is then reduced to the control of a two dimensional variable $(\tilde{q}_{1,0}(s), \tilde{q}_{1,1}(s))$. Afterwards, we conclude the proof of part *i)* of proposition 3.2.

-In Step 3, we use dynamics of $(\tilde{q}_{1,0}(s), \tilde{q}_{1,1}(s))$ to prove its transversality on $\partial V_A(s)$ (part *ii)* of proposition 3.2).

Step 1: A priori estimates of q.
From equations (27) and (28) (which are equivalent to the first equation of system (19)), we write the integral equations satisfied by $\tilde{q}_1$ and $\tilde{q}_2$:

$$\begin{aligned}
\tilde{q}_1(s) &= K_1(s,\sigma)\tilde{q}_1(\sigma) + \int_\sigma^s d\tau K_1(s,\tau)V_{1,2}(\tau)\tilde{q}_2(\tau) \\
&+ \int_\sigma^s d\tau K_1(s,\tau)\tilde{B}_1(q)d\tau + \int_\sigma^s d\tau K_1(s,\tau)\tilde{R}_1^*(\tau) \\
(36) \qquad &+ \int_\sigma^s d\tau K_1(s,\tau)\frac{d\theta}{ds}(\tau)\{\delta\tilde{\varphi}_1(\tau) + \tilde{\varphi}_2(\tau) + \delta\tilde{q}_1(\tau) + \tilde{q}_2(\tau)\} \\
\tilde{q}_2(s) &= K_2(s,\sigma)\tilde{q}_2(\sigma) + \int_\sigma^s d\tau K_2(s,\tau)V_{2,1}(\tau)\tilde{q}_1(\tau) \\
&+ \int_\sigma^s d\tau K_2(s,\tau)\tilde{B}_2(q)d\tau + \int_\sigma^s d\tau K_2(s,\tau)\tilde{R}_2^*(\tau) \\
&- \int_\sigma^s d\tau K_2(s,\tau)\frac{d\theta}{ds}(\tau)\{(1+\delta^2)\tilde{\varphi}_1(\tau) + \delta\tilde{\varphi}_2(\tau)) + (1+\delta^2)\tilde{q}_1(\tau) \\
(37) \qquad &+ \delta\tilde{q}_2(\tau)\}
\end{aligned}$$

where K_1 is the fundamental solution of $\mathcal{L}+V_{1,1}$, K_2 is the fundamental solution of $\mathcal{L}-1+V_{2,2}$, $\mathcal{L}$ is given by (30),
$B(q) = (1+i\delta)\tilde{B}_1 + i\tilde{B}_2$,
$R^*(y,s) = (1+i\delta)\tilde{R}_1^* + i\tilde{R}_2^*$, B and R^* are given by (20).

We now assume that for each $s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$. Using (36, 37), we derive new bounds on all terms in the right hand sides of (36, 37), and then on q.
In the case $\sigma = s_0$, from initial data properties, it turns out that we obtain better estimates for $s \in [s_0, s_0+\rho]$.
More precisely, we have the following lemma:

Lemma 3.3 *There exists $A_4 > 0$ such that for each $A \geq A_4$, $\rho^* > 0$, there exists $s_4(A,\rho^*) > 0$ with the following property:*
$\forall\delta \in [-1/2, 1/2]$, $\forall s_0 \geq s_4(A,\rho^*)$, $\forall\rho \leq \rho^*$, assume $\forall s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$ with $\sigma \geq s_0$.

I)$\tilde{q}_1$ estimates:
We have $\forall s \in [\sigma, \sigma+\rho]$,
i) **(main linear term)**

$$\begin{aligned}
|\alpha_{1,2}(s)| &\leq A^2\frac{\log\sigma}{s^2} + (s-\sigma)CAs^{-3}, \\
|\alpha_{1,-}(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A + e^{-(s-\sigma)^2}A^2)(1+|y|^3)s^{-2}, \\
\|\alpha_{1,e}(s)\|_{L^\infty} &\leq C(A^2e^{-\frac{(s-\sigma)}{2p}} + Ae^{(s-\sigma)})s^{-\frac{1}{2}},
\end{aligned}$$

where, as in decomposition (22),

$$K_1(s,\sigma)\tilde{q}_1(\sigma) = \alpha_1(y,s) = \sum_{m=0}^{2}\alpha_{1,m}(s)h_m(y) + \alpha_{1,-}(y,s) + \alpha_{1,e}(y,s).$$

If $\sigma = s_0$, and $q(s_0)$ satisfies (25), then

$$
\begin{aligned}
|\alpha_{1,2}(s)| &\leq \frac{\log s_0}{s^2} + CA(s-s_0)s^{-3},\\
|\alpha_{1,-}(y,s)| &\leq C(1+|y|^3)s^{-2}, \|\alpha_{1,e}(s)\|_{L^\infty} \leq C(1+e^{(s-s_0)})s^{-\frac{1}{2}}.
\end{aligned}
$$

*ii)***(interference term)**

$$
\begin{aligned}
|\iota_{1,2}(s)| &\leq C|\delta|A(s-\sigma)e^{s-\sigma}s^{-3},\\
|\iota_{1,-}(y,s)| &\leq C|\delta|A^2(s-\sigma)(1+|y|^3)s^{-2},\\
\|\iota_{1,e}(s)\|_{L^\infty} &\leq C|\delta|(A^2+e^{(s-\sigma)}A)(s-\sigma)s^{-1/2},
\end{aligned}
$$

where, as in decomposition (22), $\int_\sigma^s d\tau K_1(s,\tau)V_{1,2}(\tau)\tilde{q}_2(\tau) =$

$$\iota_1(y,s) = \sum_{m=0}^{2} \iota_{1,m}(s)h_m(y) + \iota_{1,-}(y,s) + \iota_{1,e}(y,s).$$

iii) **(nonlinear term)**

$$
\begin{aligned}
|\beta_{1,2}(s)| &\leq \frac{(s-\sigma)}{s^{3+1/2}},\\
|\beta_{1,-}(y,s)| &\leq (s-\sigma)(1+|y|^3)s^{-2-\epsilon}, \|\beta_{1,e}(s)\|_{L^\infty} \leq (s-\sigma)s^{-\frac{1}{2}-\epsilon},
\end{aligned}
$$

where $\epsilon = \epsilon(p) > 0$, *and as in (22),* $\int_\sigma^s d\tau K_1(s,\tau)\tilde{B}_1(q(\tau)) =$

$$\beta_1(y,s) = \sum_{m=0}^{2} \beta_{1,m}(s)h_m(y) + \beta_{1,-}(y,s) + \beta_{1,e}(y,s).$$

iv) **(main corrective term)**

$$
\begin{aligned}
|\gamma_{1,2}(s)| &\leq (s-\sigma)Cs^{-3},\\
|\gamma_{1,-}(y,s)| &\leq (s-\sigma)C(1+|y|^3)s^{-2}, \|\gamma_{1,e}(s)\|_{L^\infty} \leq (s-\sigma)s^{-3/4},
\end{aligned}
$$

where as in (22),

$$\int_\sigma^s d\tau K_1(s,\tau)\tilde{R}_1^*(\tau) = \gamma_1(y,s) = \sum_{m=0}^{2} \gamma_{1,m}(s)h_m(y) + \gamma_{1,-}(y,s) + \gamma_{1,e}(y,s).$$

v) **(small terms)**

$$
\begin{aligned}
|\lambda_{1,2}(s)| &\leq C(s-\sigma)s^{-3},\\
|\lambda_{1,-}(y,s)| &\leq C(s-\sigma)(1+|y|^3)s^{-3}, \|\lambda_{1,e}(s)\|_{L^\infty} \leq C(s-\sigma)s^{-3/2},
\end{aligned}
$$

where as in (22), $\int_\sigma^s d\tau K_1(s,\tau)\frac{d\theta}{ds}(\tau)\{\delta\tilde{q}_1(\tau) + \tilde{q}_2(\tau) + \delta\tilde{\varphi}_1(\tau) + \tilde{\varphi}_2(\tau)\} =$

$$\lambda_1(y,s) = \sum_{m=0}^{2} \lambda_{1,m}(s)h_m(y) + \lambda_{1,-}(y,s) + \lambda_{1,e}(y,s).$$

II)$\tilde{q}_2$ estimates:
We have $\forall s \in [\sigma, \sigma+\rho]$,
i) **(main linear term)**

$$\begin{aligned} |\alpha_{2,\perp}(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A + e^{-(s-\sigma)^2}A^2)(1+|y|^3)s^{-2}, \\ \|\alpha_{2,e}(s)\|_{L^\infty} &\leq C(A^2 e^{-\frac{(s-\sigma)}{p}} + A)s^{-\frac{1}{2}}, \end{aligned}$$

where, as in decomposition (23),

$$K_2(s,\sigma)\tilde{q}_2(\sigma) = \alpha_2(y,s) = \alpha_{2,0}(s)h_0(y) + \alpha_{2,\perp}(y,s) + \alpha_{2,e}(y,s).$$

If $\sigma = s_0$, *and* $q(s_0)$ *satisfies (25), then*

$$|\alpha_{2,\perp}(y,s)| \leq C(1+|y|^3)s^{-2}, \|\alpha_{2,e}(s)\|_{L^\infty} \leq Cs^{-\frac{1}{2}}. \tag{38}$$

ii) **(interference term)**

$$|\iota_{2,\perp}(y,s)| \leq C|\delta|A(s-\sigma)(1+|y|^3)s^{-2}, \|\iota_{2,e}(s)\|_{L^\infty} \leq C|\delta|A^2(s-\sigma)s^{-1/2},$$

where as in (23), $\int_\sigma^s d\tau K_2(s,\tau)V_{2,1}(\tau)\tilde{q}_1(\tau) =$

$$\iota_2(y,s) = \iota_{2,0}(s)h_0(y) + \iota_{2,\perp}(y,s) + \iota_{2,e}(y,s).$$

iii) **(nonlinear term)**

$$|\beta_{2,\perp}(y,s)| \quad \leq \quad (s-\sigma)(1+|y|^3)s^{-2-\epsilon}, \|\beta_{2,e}(s)\|_{L^\infty} \leq (s-\sigma)s^{-\frac{1}{2}-\epsilon},$$

where $\epsilon = \epsilon(p) > 0$, *and as in (23),*

$$\int_\sigma^s d\tau K_2(s,\tau)\tilde{B}_2(q(\tau)) = \beta_2(y,s) = \beta_{2,0}(s)h_0(y) + \beta_{2,\perp}(y,s) + \beta_{2,e}(y,s).$$

iv) **(main corrective term)**

$$|\gamma_{2,\perp}(y,s)| \quad \leq \quad Cs^{-2}(s-\sigma)(1+|y|^3), \|\gamma_{2,e}(s)\|_{L^\infty} \leq (s-\sigma)s^{-3/4},$$

where as in (23),

$$\int_\sigma^s d\tau K_2(s,\tau)\tilde{R}_2^*(\tau) = \gamma_2(y,s) = \gamma_{2,0}(s)h_m(y) + \gamma_{2,\perp}(y,s) + \gamma_{2,e}(y,s).$$

v) **(small terms)**

$$|\lambda_{2,\perp}(y,s)| \quad \leq \quad C(s-\sigma)(1+|y|^3)s^{-2}, \|\lambda_{2,e}(s)\|_{L^\infty} \leq C(s-\sigma)s^{-2},$$

where $\int_\sigma^s d\tau K_2(s,\tau)\frac{d\theta}{ds}(\tau)\{-\delta\tilde{q}_2(\tau) - (1+\delta^2)\tilde{q}_1(\tau) - \delta\tilde{\varphi}_2(\tau) - (1+\delta^2)\tilde{\varphi}_1(\tau)\} =$
$\lambda_2(y,s) = \lambda_{2,0}(s)h_0(y) + \lambda_{2,\perp}(y,s) + \lambda_{2,e}(y,s)$, *as in (23).*

Proof: see appendix B .

Step 2: Lemma 3.3 implies *i*) of proposition 3.2

Here, we derive *i*) of proposition 3.2 from lemma 3.3. We follow the method used in [18] to prove proposition 3.4 starting from lemma 3.12. Indeed, from integral equations (36, 37) and lemma 3.3, we derive new bounds on $\tilde{q}_{1,2}(s)$,

$\tilde{q}_{1,-}(y,s)$, $\tilde{q}_{1,e}(y,s)$, $\tilde{q}_{2,\perp}(s)$ and $\tilde{q}_{2,e}(y,s)$, assuming that $\forall s \in [\sigma, \sigma+\rho]$, $q(s) \in V_A(s)$, for $\rho \le \rho^*$ and $\sigma \ge s_0 \ge s_4(A,\rho^*)$. The key estimate is to show that for $s = \sigma + \rho$ (or $s \in [\sigma, \sigma+\rho]$ if $\sigma = s_0$), these new bounds are better than those defining $V_A(s)$, provided that $\rho \le \rho^*(A)$.

Comparing lemma 3.3 here and lemma 3.4 in [18], we see that we have additional terms:

-Interference terms $Iii)$ and $IIii)$,

- Small terms $Iv)$ and $IIv)$.

If we try to adapt the proof of proposition 3.4 of [18] in order to prove a similar result, we see that the introduction of small terms does not change anything to the proof, since they are

either of lower order, if compared for example with linear terms (speaking in terms of power of s): $\lambda_{1.-}$, $\lambda_{1,e}$ and $\lambda_{2,e}$,

or of the same order, but with a "small" coefficient (compared with A): $\lambda_{1,2}$ and $\lambda_{2,\perp}$.

This is not the case of interference terms $Ii)$ and $IIi)$, which have a critical growth in terms of power of s. But recalling that in the mentioned proof in [18], we have $(s-\sigma) \le \rho \le \rho^* \le \log \frac{A}{C^*}$, if we assume that:

$C|\delta| A \log \frac{A}{C^*} e^{\log \frac{A}{C^*}} \le 1$ (Cf $\iota_{1,2}$), $C|\delta| A^2 \log \frac{A}{C^*} \le \frac{A}{4}$ (Cf $\iota_{1,-}$), $C|\delta|(A^2 + e^{\log \frac{A}{C^*}} A) \log \frac{A}{C^*} \le \frac{A^2}{4}$ (Cf $\iota_{1,e}$), $C|\delta| A \log \frac{A}{C^*} \le \frac{A}{4}$ (Cf $\iota_{2,\perp}$) and

$C|\delta| A^2 \log \frac{A}{C^*} \le \frac{A^2}{4}$ (Cf $\iota_{2,e}$),

which is possible if $|\delta| \le \delta_5(A)$, with $\delta_5(A) > 0$, then all these terms, while remaining with critical growth, have a reasonable coefficient (1, $\frac{A}{4}$ or $\frac{A^2}{4}$).

Therefore, adapting the proof of Proposition 3.4 in [18] for $|\delta| \le \delta_5(A)$, we prove a similar proposition:

Proposition 3.5 *There exists $A_5 > 0$ such that for each $A \ge A_5$, there exists $\delta_5(A) > 0$, $s_5(A) > 0$ such that for each $\delta \in [-\delta_5, \delta_5]$, $s_0 \ge s_5(A)$, we have the following property:*
-if (d_0, d_1) is chosen so that $(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1}(s_0)) \in \hat{V}_A(s_0)$, and,
-if for $s_1 \ge s_0$, we have $\forall s \in [s_0, s_1]$, $q(s) \in V_A(s)$,
then $\forall s \in [s_0, s_1]$, $|\tilde{q}_{1,2}(s)| \le A^2 s^{-2} \log s - s^{-3}$, $|\tilde{q}_{1,-}(y,s)| \le \frac{A}{2}(1+|y|^3)s^{-2}$, $\|\tilde{q}_{1,e}(s)\|_{L^\infty} \le \frac{A^2}{2\sqrt{s}}$, $|\tilde{q}_{2,\perp}(y,s)| \le \frac{A}{2}(1+|y|^3)s^{-2}$, $\|\tilde{q}_{2,e}(s)\|_{L^\infty} \le \frac{A^2}{2\sqrt{s}}$.

By definition of (q, θ) (Cf system (19)), we have $\tilde{q}_{2,0}(s) = 0$. If in addition $q(s_1) \in \partial V_A(s_1)$, we see from definition 3.1 of $V_A(s)$ that the first two components of $q(s_1)$, namely $\tilde{q}_{1,0}(s_1)$ and $\tilde{q}_{1,1}(s_1)$ are in $\partial \hat{V}_A(s_1)$. This concludes the proof of part $i)$ of proposition 3.2.

Step 3: Transversality property of $(\tilde{q}_{1,0}(s_1), \tilde{q}_{1,1}(s_1))$ on $\partial \hat{V}_A(s_1)$

To prove part $ii)$ of proposition 3.2, we show that for each $m \in \{0,1\}$, for each $\epsilon \in \{-1,1\}$, if $\tilde{q}_{1,m}(s_1) = \epsilon \frac{A}{s_1^2}$, then $\frac{d\tilde{q}_{1,m}}{ds}(s_1)$ has the opposite sign of $\frac{d}{ds}(\frac{\epsilon A}{s^2})(s_1)$ so that $(\tilde{q}_{1,0}, \tilde{q}_{1,1})$ actually leaves $\hat{V}_A$ at s_1 for $s_1 \ge s_0$ where s_0 will be large. Now, let us compute $\frac{d\tilde{q}_{1,0}}{ds}(s_1)$ and $\frac{d\tilde{q}_{1,1}}{ds}(s_1)$ for $q(s_1) \in V_A(s_1)$ and $(\tilde{q}_{1,0}(s_1), \tilde{q}_{1,1}(s_1)) \in \partial \hat{V}_A(s_1)$. First, we note that in this case, $\|q(s_1)\|_{L^\infty} \le \frac{CA^2}{\sqrt{s_1}}$

and $|q_b(y,s_1)| \le CA^2\frac{\log s_1}{s_1^2}(1+|y|^3)$ (Provided $A \ge 1$). Below, $O(l)$ stands for a quantity whose absolute value is bounded precisely by l and not Cl.
For $m \in \{0,1\}$, we derive from equation (27) and (21): $\int d\mu\chi(s_1)\frac{\partial \tilde{q}_1}{\partial s}k_m =$

$$\int d\mu\chi(s_1)\mathcal{L}\tilde{q}_1 k_m + \int d\mu\chi(s_1)\{V_{1,1}\tilde{q}_1 + V_{1,2}\tilde{q}_2\}k_m + \int d\mu\chi(s_1)\tilde{B}_1(q)k_m$$

$+\int d\mu\chi(s_1)\tilde{R}_1^*(s_1)k_m + \int d\mu\chi(s_1)\frac{d\theta}{ds}(s_1)\{\delta\tilde{q}_1 + \tilde{q}_2 + \delta\tilde{\varphi}_1 + \tilde{\varphi}_2\}k_m$, where $k_m = h_m/\|h_m\|^2_{L^2(\mathbb{R},\mathbb{R},d\mu)}$ (see (21)).

We now estimate each term of this identity:

a) $|\int d\mu\chi(s_1)\frac{\partial\tilde{q}_1}{\partial s}k_m - \frac{d\tilde{q}_{1,m}}{ds}| = |\int d\mu\frac{d\chi}{ds}\tilde{q}_1 k_m| \le \int d\mu|\frac{d\chi}{ds}|\frac{CA^2}{\sqrt{s_1}}|k_m| \le Ce^{-s_1}$ if $s_0 \ge s_3(A)$.

b) Since $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{R},d\mu)$, we write

$$\int d\mu\chi(s_1)\mathcal{L}\tilde{q}_1 k_m = \int d\mu\mathcal{L}(\chi(s_1)k_m)\tilde{q}_1.$$

Using $\mathcal{L}(\chi(s_1)k_m) = (1-\frac{m}{2})\chi(s_1)k_m + \frac{\partial^2\chi}{\partial s^2}k_m + \frac{\partial\chi}{\partial y}(2\frac{\partial k_m}{\partial y} - \frac{y}{2}k_m)$,
we obtain $\int d\mu\chi(s_1)\mathcal{L}\tilde{q}_1 k_m = (1-\frac{m}{2})\tilde{q}_{1,m}(s_1) + O(CAe^{-s_1})$.

c) We have $\forall y \in \mathbb{R}, |V_{i,j}(y,s)| \le \frac{C}{s}(1+|y|^2)$. Therefore, $|\int d\mu\chi(s_1)\{V_{1,1}\tilde{q}_1 + V_{1,2}\tilde{q}_2\}k_m| \le \int d\mu Cs_1^{-1}(1+|y|^2)CA^2 s_1^{-2}\log s_1|k_m| \le CA^2 s_1^{-3}\log s_1$

d) A standard Taylor expansion combined with the definition of V_A shows that $|\chi(y,s_1)B(q(y,s_1))| \le C|q|^2 \le C(|q_b|^2+|q_e|^2) \le \frac{CA^4(\log s_1)^2}{s_1^4}(1+|y|^3)^2 + 1_{\{|y|\ge K\sqrt{s_1}\}}(y)\frac{A^2}{\sqrt{s_1}}$. Thus, $|\int d\mu\chi(s_1)\tilde{B}_1(q)k_m| \le \frac{CA^4(\log s_1)^2}{s_1^4} + Ce^{-s_1}$.

e) From lemma B.5 in appendix B, we have $|\int d\mu\chi(s_1)\tilde{R}_1^*(s_1)k_m| \le \frac{C(p)}{s_1^2}$ (Actually it is equal to 0 if $m=1$).

f) From lemma 3.1, we have $|\frac{d\theta}{ds}(s_1)| \le Cs_1^{-2}$. Hence, $|\int d\mu\chi(s_1)\frac{d\theta}{ds}(s_1)\{\delta\tilde{q}_1 + \tilde{q}_2 + \delta\tilde{\varphi}_1 + \tilde{\varphi}_2)k_m| \le Cs_1^{-2}$.

Putting together the estimates a) to f), we obtain

$$\frac{d\tilde{q}_{1,m}}{ds}(s_1) = (1-\frac{m}{2})\frac{\epsilon A}{s_1^2} + O(\frac{C(p)}{s_1^2}) + O(CA^4\frac{\log s_1}{s_1^3})$$

whenever $\tilde{q}_{1,m}(s_1) = \frac{\epsilon A}{s_1^2}$. Let us now fix $A \ge 2C(p)$, and then we take $s_1(A)$ larger so that for $s_0 \ge s_1(A)$, $\forall s \ge s_0$, $\frac{C(p)}{s^2} + O(CA^4\frac{\log s}{s^3}) \le \frac{3C(p)}{2s^2}$. Hence, if $\epsilon = -1$, $\frac{d\tilde{q}_{1,m}}{ds}(s_1) < 0$, if $\epsilon = 1$, $\frac{d\tilde{q}_{1,m}}{ds}(s_1) > 0$. This concludes the proof of part *ii)* of proposition 3.2. It also concludes the proof of part *ii)* of Proposition 1, and then the proof of Theorem 1. ∎

4 Blow-up profile of $u(t)$ solution of (2) near blow-up point

We prove in this section part *iii)* of Proposition 1.

We consider $u(t)$ solution of (1) constructed in section 3, which blows-up in finite time $T > 0$ at only one blow-up point: 0. We know from section 3 that:

$$(39)\quad \sup_{z\in\mathbb{R}}|(T-t)^{\frac{1+i\delta}{p-1}}u(z\sqrt{(T-t)|\log(T-t)|},t) - f(z)| \le \frac{C}{\sqrt{|\log(T-t)|}}$$

with

(40) $$f(z) = (p-1+\frac{(p-1)^2}{4(p-\delta^2)}|z|^2)^{-\frac{1+i\delta}{p-1}}.$$

Adapting the techniques used by Merle in [17] to equation (1), we derive the existence of a profile $u_* \in \mathcal{C}(\mathbb{R}\backslash\{0\}, \mathbb{C})$ such that $u(x,t) \to u_*(x)$ as $t \to T$ uniformly on compact subsets of $\mathbb{R}\backslash\{0\}$. We want to find an equivalent function for u_* near the blow-up point: 0.

For this purpose, we define for each $t \in [0,T)$, a rescaled version of $u(t)$:

(41) $$v(t,\xi,\tau) = (T-t)^{\frac{1+i\delta}{p-1}} u(\xi\sqrt{T-t}, t+(T-t)\tau)$$

where $\xi \in \mathbb{R}$, $\tau \in [-\frac{t}{T-t}, 1) \subset [0,1)$. From equation (1), we see that $v(t,\xi,\tau)$ satisfies the same equation as $u(t,x)$:

(42) $$\forall \tau \in [-\frac{t}{T-t}, 1), \frac{\partial v}{\partial \tau} = \Delta_\xi v + (1+i\delta)|v|^{p-1}v.$$

Stated in terms of $v(t)$, (39) becomes:

(43) $$\sup_{\xi\in\mathbb{R}} |(1-\tau)^{\frac{1+i\delta}{p-1}} v(t,\xi,\tau) - f(\frac{\xi}{\sqrt{(1-\tau)|\log\{(1-\tau)(T-t)\}|}})| \leq$$
$$\frac{C}{\sqrt{|\log\{(T-t)(1-\tau)\}|}}.$$

We proceed in two steps:

- first, we consider $r > 0$ and estimate $v(t,\xi,\tau)$ and its derivatives locally near $\xi(r,t) \in \mathbb{R}$ satisfying $|\xi(r,t)| = r\sqrt{|\log(T-t)|}$. We show that $v(t,\xi,\tau)$ is bounded, and that it does not vary much for $|\xi - \xi(r,t)|$ bounded and $\tau \in [0,1]$,

- then, we can identify $v(t,\xi,0)$ (approximated by (43)) and $v(t,\xi,1)$. For each $x \in \mathbb{R}\backslash\{0\}$, we write $|x|$ as $|\xi(r,t)|\sqrt{(T-t)} = r\sqrt{(T-t)|\log(T-t)|}$ for some $r > 0$ and $t < T$ and combine this identification with (41) to get the equivalent of $u_*(x)$ for $x \to 0$:

(44) $$u_*(x) \sim \left[\frac{8(p-\delta^2)|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1+i\delta}{p-1}}.$$

For simplicity, we omit t in the notation and write $v(\xi,\tau)$ for $v(t,\xi,\tau)$, $\xi(r)$ for $\xi(r,t)$.

Part I: Estimate for v near $r\sqrt{|\log(T-t)|}$

From (41), v blows-up at time $\tau = 1$ at only one blow-up point: 0. Using (43) and a lower bound shown by Giga and Kohn in [11] on blow-up rate for v, we derive a local bound on v for $\tau \in [0,1)$, $|\xi - \xi(r)|$ bounded, independent from r and t. Using classical parabolic theory and the fact that v depends in a certain sense only on τ for $|\tau|$ small, we show that v actually does not depend much on $\tau \in [0,1)$ for $|\xi - \xi(r)|$ bounded.

Proposition 4.1 (Estimate on $\frac{\partial v}{\partial \tau}(\xi(r),\tau)$) *There exists $r_1 > 0$ such that $\forall r \geq r_1$, $\exists t_1(r) < T$ such that $\forall t \in [t_1(r), T)$, $\forall \tau \in [0,1)$, $|\frac{\partial v}{\partial \tau}(\xi(r),\tau)| \leq C|f(r)|^p$.*

Proof:

Step 1: Local bounds on v near $\xi(r)$ for $\tau \in [-1/2, 1)$
We crucially use a lower bound on blow-up rate for v established by Giga and Kohn in [11] to show that $|v|$ is bounded for ξ near $\xi(r)$ and $\tau \in [-1/2, 1)$.

Lemma 4.1 (Lower bound on blow-up rate for v) .
i) **(Giga-Kohn)** *There exists $\epsilon = \epsilon(p, \delta, N) > 0$ with the following property: If for $|\xi - \xi(r)| \le 3\sqrt{|\log(T-t)|}$, $\tau \in [-1/2, 1)$*

$$(1-\tau)^{\frac{1}{p-1}}|v(\xi,\tau)| \le \epsilon,$$

then $\forall \xi \in \mathbb{R}$ with $|\xi - \xi(r)| \le 2\sqrt{|\log(T-t)|}$, $\forall \tau \in [-1/2, 1)$, $|v(\xi, \tau)| \le C$.

ii) There exists $r_2 > 0$ such that $\forall r \ge r_2$, $\exists t_2(r) < T$ such that $\forall t \in [t_2(r), T)$, if $|\xi - \xi(r)| \le 2\sqrt{|\log(T-t)|}$, $\tau \in [-1/2, 1)$ then

$$|v(\xi, \tau)| \le C.$$

Proof:
i) follows immediately from Theorem 2.1 in [11]. *ii)* is a direct consequence of *i)* and estimate (43). Indeed, if $|\xi - \xi(r)| \le 3\sqrt{|\log(T-t)|}$ and $\tau \in [-1/2, 1)$, then we have by (43) $(1-\tau)^{\frac{1}{p-1}}|v(\xi,\tau)| \le C|f(r)| + C|\log(T-t)|^{-1/2}$. ■

Step 2: Local bound on $\frac{\partial v}{\partial \tau}(\xi, \tau)$ near $\xi(r)$ for $\tau \in [0, 1)$
- $\tau = 0$: From a parabolic estimate and (43) considered for $\tau \le 0$, we have for $|\xi - \xi(r)| \le \sqrt{|\log(T-t)|}$:

$$\left|\frac{\partial^2 v}{\partial \xi^2}(\xi, 0) - \frac{1}{|\log(T-t)|}\frac{\partial^2 f}{\partial z^2}\left(\frac{\xi}{\sqrt{|\log(T-t)|}}\right)\right| \le \frac{C}{\sqrt{|\log(T-t)|}}.$$

Hence, from (42), we have for $r \ge r_3$, $t \ge t_3(r)$, $|\xi - \xi(r)| \le \sqrt{|\log(T-t)|}$: $|\frac{\partial v}{\partial \tau}(\xi, 0)| \le C|f(r)|^p$.

- $\tau \in [0, 1)$: We use the equation satisfied by $\frac{\partial v}{\partial \tau}$ and standard tools of localization and local estimates with the semi-group $e^{\tau\Delta}$ to conclude. Indeed, if $z(\xi, \tau) = |\frac{\partial v}{\partial \tau}|^2$, it follows from equation (42) and *ii)* of lemma 4.1 that $\forall \tau \in [0, 1)$, $\forall \xi \in \mathbb{R}$ with $|\xi - \xi(r)| \le \sqrt{|\log(T-t)|}$, $\frac{\partial z}{\partial \tau} \le \Delta z + Mz$, where $M = M(p, \delta, N)$.
We can consider $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\phi(\xi) = 0$ if $|\xi - \xi(r)| \ge \sqrt{|\log(T-t)|}$, $0 \le \phi \le 1$, $\phi(\xi) = 1$ if $|\xi - \xi(r)| \le \sqrt{|\log(T-t)|}/2$, and $|\nabla\phi| + |\Delta\phi| \le C$.
If $w(\xi, \tau) = e^{-\tau M}\phi(\xi)z(\xi, \tau)$, then w satisfies:
$\frac{\partial w}{\partial \tau} \le \Delta w + e^{-\tau M}(-z\Delta\phi + 2\nabla z.\nabla\phi)$ and $\forall \xi \in \mathbb{R}$, $|w(\xi, 0)| \le C|f(r)|^{2p}$.
If $\tau \in [0, 1)$, then

$$\begin{aligned} w(\xi(r), \tau) &\le (e^{\tau\Delta}w(0))(\xi(r), \tau) \\ &+ \int_0^\tau \frac{d\sigma}{(4\pi(\tau-\sigma))^{1/2}} \int dx e^{-\frac{|x-\xi(r)|^2}{4(\tau-\sigma)}} (z|\Delta\phi| + 2|\nabla z||\nabla\phi|)(x, \sigma) \\ &\le C|f(r)|^{2p} + \int_0^\tau \frac{d\sigma}{(4\pi(\tau-\sigma))^{1/2}} \int dx e^{-\frac{|\log(T-t)|/4}{8(\tau-\sigma)}} e^{-\frac{|x-\xi(r)|^2}{8(\tau-\sigma)}} C \end{aligned}$$

(lemma 4.1 *ii*) implies by parabolic regularity that for $r \geq r_2$, $t \geq t_2(r)$, $(z|\Delta\phi| + 2|\nabla z||\nabla\phi|)(x,\sigma) \leq C$, for $\sigma \in [0,1)$ and $|x - \xi(r)| \leq \sqrt{|\log(T-t)|})$.

Therefore, $w(\xi(r),\tau) \leq C|f(r)|^{2p} + e^{-|\log(T-t)|}$. If $t \geq t_4(r)$, then $w(\xi,\tau) \leq C|f(r)|^{2p}$, which implies $\forall\tau \in [0,1)$, $|\frac{\partial v}{\partial\tau}(\xi(r),\tau)| \leq C|f(r)|^p$.

Taking $r_1 = \max(r_2, r_3)$ and $t_1(r) = \max(t_2(r), t_3(r), t_4(r))$ concludes the proof. ■

Part II: Conclusion of the proof

For each $r \geq r_1$ and each $x \in \mathbb{R}\backslash\{0\}$ small enough, we define $t(r,x) \in [0,T)$ by

$$|x| = |\xi(r)|\sqrt{T-t} = r\sqrt{(T - t(r,x))|\log(T - t(r,x))|}. \tag{45}$$

Applying proposition 4.1 to $v(t(r,x))$, we estimate the difference between $u_*(x)$ and $u(x, t(r,x))$ and then between $u_*(x)$ and $f(r)$. Then, by simple asymptotic calculation, we reach the equivalent (44).

Lemma 4.2 (A first estimate on the profile $u_*(x)$) *$\forall r \geq r_1$, $\exists R_2(r) > 0$ such that $\forall x \in \mathbb{R}$ with $0 < |x| < R_2$*

$$|(T - t(r,x))^{\frac{1+i\delta}{p-1}} u_*(x) - f(r)| \leq C|f(r)|^p,$$

where $t(r,x)$ is uniquely determined by (45).

Proof:

Using proposition 4.1 and (43), we write for $r \geq r_1$, $t \geq t_1(r)$: $\forall\tau \in [0,1)$ $|v(\xi(r),\tau) - f(r)| \leq |v(\xi(r),\tau) - v(\xi(r),0)| + |v(\xi(r),0) - f(r)| \leq C|f(r)|^p + C|\log(T-t)|^{-1/2}$.

Stated in terms of u, this gives: $\forall\tau \in [0,1)$

$$|(T-t)^{\frac{1+i\delta}{p-1}} u(\xi(r)\sqrt{T-t}, t + (T-t)\tau) - f(r)| \tag{46}$$

$\leq C|f(r)|^p + C|\log(T-t)|^{-1/2}$.

From this estimate, we derive $R_2(r) > 0$ such that $\forall x \in \mathbb{R}$ with $0 < |x| < R_2$, we have: $\forall\tau \in [0,1)$
$|(T - t(r,x))^{\frac{1+i\delta}{p-1}} u(x, t(r,x) + (T - t(r,x))\tau) - f(r)| \leq C|f(r)|^p$,
where $t(r,x)$ is given by (45). If we let τ go to 1, we have the conclusion of lemma 4.2. ■

Now, we conclude the proof of estimate (44). For this purpose, we consider an arbitrary $\epsilon > 0$ and look for $R_\epsilon > 0$ such that for $0 < |x| < R_\epsilon$,

$$\left| \left[\frac{|x|^2}{-\log|x|}\right]^{\frac{1+i\delta}{p-1}} u_*(x) - \left[\frac{8(p-\delta^2)}{(p-1)^2}\right]^{\frac{1+i\delta}{p-1}} \right| \leq \epsilon.$$

If we consider an arbitrary $r \geq r_1$, then by lemma 4.2, we have for $0 < |x| < R_2$

$$\begin{aligned} &\left| \left[\frac{|x|^2}{-\log|x|}\right]^{\frac{1+i\delta}{p-1}} u_*(x) - \left[\frac{8(p-\delta^2)}{(p-1)^2}\right]^{\frac{1+i\delta}{p-1}} \right| \\ \leq \; &\left| \left[\frac{|x|^2}{-\log|x|}\right]^{\frac{1+i\delta}{p-1}} - [2r^2(T - t(r,x))]^{\frac{1+i\delta}{p-1}} \right| . |u_*(x)| \end{aligned} \tag{47}$$

$$
\begin{aligned}
&+ \quad [2r^2]^{\frac{1}{p-1}} |(T - t(r,x))^{\frac{1+i\delta}{p-1}} u_*(x) - f(r)| \\
&+ \quad |[2r^2]^{\frac{1+i\delta}{p-1}} f(r) - \left[\frac{8(p-\delta^2)}{(p-1)^2}\right]^{\frac{1+i\delta}{p-1}} |
\end{aligned}
$$

We fix $r(\epsilon) \geq r_1$ such that $|[2r^2]^{\frac{1+i\delta}{p-1}} f(r) - \left[\frac{8(p-\delta^2)}{(p-1)^2}\right]^{\frac{1+i\delta}{p-1}}| \leq \epsilon$ and $|f(r)|^{p-1} \leq \epsilon$.

From (45), we have

$$
\frac{|x|^2}{-\log|x|} = 2r^2(T - t(r,x)) \frac{\log(T - t(r,x))}{\log(T - t(r,x)) + \log|\log(T - t(r,x))| + 2\log r}.
$$

Let $R_\epsilon > 0$ sufficiently small and smaller than $R_2(r(\epsilon))$ such that for $0 < |x| < R_\epsilon$

$$
|\left[\frac{|x|^2}{-\log|x|}\right]^{\frac{1+i\delta}{p-1}} - [2r^2(T - t(r,x))]^{\frac{1+i\delta}{p-1}}| \leq \epsilon [2r^2(T - t(r,x))]^{\frac{1}{p-1}}.
$$

Hence, for $0 < |x| < R_\epsilon$, we have from (47): $|[\frac{|x|^2}{-\log|x|}]^{\frac{1+i\delta}{p-1}} u_*(x) - [\frac{8(p-\delta^2)}{(p-1)^2}]^{\frac{1+i\delta}{p-1}}|$
$\leq \epsilon[2r^2(T - t(r,x))]^{\frac{1}{p-1}} |u_*(x)| + C\epsilon r^{\frac{2}{p-1}} |f(r)| + \epsilon$
$\leq C\epsilon r^{\frac{2}{p-1}} |f(r)|(1 + C\epsilon) + C\epsilon$ (use lemma 4.2 and $|f(r)|^{p-1} \leq \epsilon$)
$\leq C\epsilon$. This concludes the proof of part *iii*) of Proposition 1.

5 Generalization and comments

As a first application of the techniques in previous sections, we have the following stability result concerning the behavior described in Proposition 1:

Theorem 3 (Stability with respect to initial data of the profile (4)) *Let $\delta \in (-\delta_1, \delta_1)$ where $\delta_1 > 0$ and consider $\hat{u}_0$ initial data constructed in Proposition 1. Let $\hat{u}(t)$ be the solution of equation (1) with initial data $\hat{u}_0$, $\hat{T}$ its blow-up time and $\hat{a}$ its blow-up point.*

Then there exists a neighborhood $\mathcal{V}$ of $\hat{u}_0$ in H with the following properties: For each $u_0 \in \mathcal{V}$, $u(t)$ blows-up in finite time $T = T(u_0)$ at one single point $a = a(u_0)$, where $u(t)$ is the solution of equation (1) with initial data u_0. Moreover, $u(t)$ approaches the profiles (6) and (7) near (T, a) similarly as $\hat{u}(t)$ does near $(\hat{T}, \hat{a})$.

The proof of this theorem relies strongly on the techniques developed in sections 2, 3 and 4. We give just the key ideas of the proof.

Consider initial data u_0 in a neighborhood of $\hat{u}_0$ and $u(t)$ the corresponding solution of (1). Then, for each (T, a) near $(\hat{T}, \hat{a})$, we introduce as in section 2 a two-parameter group acting on $u(t)$:

$$
(T, a) \to (q(T, a, y, s), \theta(T, a, s))
$$

where

$$
\begin{cases}
q(T, a, y, s) &= w(T, a, y, s) - \varphi(y, s) \\
\tilde{q}_{2,0}(s) &= 0,
\end{cases}
$$

$w(T,a)$ is defined similarly as in (9) by

$$\begin{aligned} y &= \tfrac{x-a}{\sqrt{T-t}} \\ s &= -\log(T-t) \\ w(y,s) &= (T-t)^{\frac{1+i\delta}{p-1}} u(x,t), \end{aligned}$$

and φ is given by (16).

Therefore, our problem reduces to searching a parameter $(T(u_0), a(u_0))$ such that

$$\forall s \geq s_0, q(T,a,s) \in V_A(s) \tag{48}$$

for some $s_0 > 0$ and $A > 0$ (see definition 3.1). Indeed, $T(u_0)$ and $a(u_0)$ will be shown then to be respectively the blow-up time and point of $u(t)$. Moreover, we derive directly form (48) an estimate analogous to (6) and then, by the techniques of section 4, an other estimate analogous to (7).

By uniform a priori estimates analogous to proposition 3.2, we reduce this problem to a finite dimensional one. We solve it using a non-degeneration property of the two-parameter group acting on $\hat{u}(t)$ itself (see [18] for similar argument). Hence, we reach the conclusion of Theorem 3.

The proof used for equation (1) applies in a more general case:
consider the following vector-valued heat equation:

$$\frac{du}{dt} = \Delta u + |u|^{p-1}u + G(u), u(x,0) = u_0(x) \tag{49}$$

where

1) $u(t) : x \in \mathbb{R}^N \to \mathbb{R}^M$, $p \in (1,+\infty)$, $p < (N+2)/(N-2)$ if $N \geq 3$,

2) $G : \mathbb{R}^M \to \mathbb{R}^M$ is a perturbation of $|u|^{p-1}u$ satisfying: $G(u) = G_1(|u|^2)u$, $|G(u)| \leq C|u|^r$, $|G(\lambda u_1) - G(\lambda u_2)| \leq C\lambda^r|u_1 - u_2|$ for $|u_1|, |u_2| \leq 1$, $\lambda \geq 1$, $r \in [1,p)$, $G_1 : \mathbb{R}^+ \to \mathbb{R}^+$, G needs not be a gradient,

3) $u_0 \in H = W^{1,p+1}(\mathbb{R}^N, \mathbb{R}^M) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^M)$.

Using the same techniques as in the case $M = 2$ (equation (1) with $\delta = 0$), we show the following blow-up result for equation (49):

Theorem 2: (Existence of a blow-up solution for equation (49))
There exist initial data u_0 such that equation (49) has a blow-up solution.

This Theorem is a direct consequence of the following proposition which describes more precisely the behavior of $u(t)$ near blow-up. Indeed, after a time dependent scaling, $u(t)$ approaches a universal profile

$$(p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}\omega, \tag{50}$$

when $t \to T$, where $\omega \in S^{M-1}$. In fact, we have the more precise result:

Proposition 2 (Existence of a blow-up solution for equation (49) with the profile (50))

There exists $T_0 > 0$ such that for each $T \in (0, T_0]$, for each $a \in \mathbb{R}^N$, for each $\omega \in S^{M-1}$, there exist initial data u_0 such that equation (49) has a blow-up solution $u(x,t)$ on $\mathbb{R}^N \times [0,T)$ which blows-up in finite time T at only one blow-up point: a. Moreover,

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) = f(z)\omega \tag{51}$$

uniformly in $z \in \mathbb{R}^N$, with

$$f(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}. \tag{52}$$

Remark: Structural stability: In [18], a particular version of this Proposition was shown in the case $M = 1$ and $G = 0$ (without perturbation): Single point blow-up and a blow-up profile (52). There, this result was shown to be stable with respect to perturbations in initial data. With proposition 2, the blow-up solution constructed in [18] is shown to be *structurally stable* in a certain class of functions, since this solution behaves in the same way when we take a non zero G and consider a higher dimension ($M \geq 2$): we still have single point blow-up with the same scalar profile (52).

A Appendix: A blow-up result for $\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u$ $+i|u|^{q-1}u$ on bounded domain for q small

We consider the complex-valued heat equation (3):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + |u|^{p-1}u + i|u|^{q-1}u \\ u_{|\partial\Omega} &= 0, \end{aligned} \tag{53}$$

where $u(t) : \Omega \to \mathbb{C}$, Ω is a bounded domain of $\mathbb{R}^N$, $p \in (1, +\infty)$, $p < (N+2)/(N-2)$ if $N \geq 3$, and $q > 1$.

Proposition A.1 (Existence of blow-up solutions for equation (53)) *Assume $1 < q < (p+1)/2$. There exists $A(\Omega, p, q) > 0$ such that for each $u_0 \in H_0^1(\Omega)$ with $\|u_0\|_{L^2(\Omega)} \geq A$ and $E(u_0) \leq 0$ where*

$$E(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1} dx, \tag{54}$$

equation (53) with initial data u_0 has a unique solution $u \in \mathcal{C}([0,T), H_0^1(\Omega))$ with $0 < T < +\infty$, which blows-up in $H_0^1(\Omega)$ as $t \to T$.

Proof:

From classical theory, we know that if $1 < q \leq p$ and $u_0 \in H_0^1(\Omega)$, then equation (53) with initial data u_0 has a unique solution defined on $[0,T)$ with $T = T_{u_0} \in (0, +\infty]$ and $u \in \mathcal{C}([0,T), H_0^1(\Omega))$. Moreover, if $T < +\infty$, then $u(t)$ blows-up in $H_0^1(\Omega)$ as $t \to T$.

Hence, proposition A.1 will be proved if we show that for $1 < q < (p+1)/2$, $\|u_0\|^2_{L^2(\Omega)} \geq A$ (to be chosen later) and $E(u_0) \leq 0$, we have $T_{u_0} < +\infty$.

We proceed as follows: first we give estimates on $u(t)$ for $t \in [0,T)$, then we use a blow-up result for an integral inequality to conclude.

Lemma A.1 (Estimate for $u(t)$, solution of (53)) *If* $z(t) = (\int_\Omega |u(x,t)|^{p+1}dx)^{2/(p+1)}$, *then* $\forall t \in [0,T)$,

$$z(t) \geq c_1 A^2 + c_2 \int_0^t d\sigma z(\sigma)^{(p+1)/2} - c_3 \int_0^t d\sigma \int_0^\sigma ds z(s)^q \tag{55}$$

where $c_1 = c_1(\Omega, p) > 0$, $c_2 = c_2(\Omega, p) > 0$ *and* $c_3 = c_3(\Omega, p, q) > 0$.

Proof:

For simplicity, we omit x, Ω and dx in following expressions of the type $\int_\Omega |u(x,t)|^2 dx$.

From (54), $\frac{d}{dt}E(u(t)) = \Re(-\int \bar{u}_t(t)\Delta u(t) - \int |u(t)|^{p-1}u(t)\bar{u}_t(t))$.

From (53), $\frac{d}{dt}E(u(t)) = \Re(-\int \bar{u}_t(t)u_t(t) + i\int |u(t)|^{q-1}u(t)\bar{u}_t(t))$
$\leq -\int |u_t(t)|^2 + \int |u(t)|^q |u_t(t)|$
$\leq -\int |u_t(t)|^2 + \frac{1}{2}(\int |u_t(t)|^2 + \int |u(t)|^{2q})$ (Cauchy Schwartz),
$\leq -\frac{1}{2}\int |u_t(t)|^2 + c_4(\Omega,p,q)(\int |u(t)|^{p+1})^{2q/(p+1)}$ (Hölder). Integrating this inequality and using $E(u_0) \leq 0$ gives

$$E(u(t)) \leq c_4(\Omega,p,q)\int_0^t ds(\int |u(s)|^{p+1})^{2q/(p+1)}. \tag{56}$$

Now, if we multiply equation (53) by $\bar{u}(t)$ and take the real part, we obtain using expression (54)

$$\frac{d}{dt}\int |u(t)|^2 = -4E(u(t)) + \frac{p-1}{p+1}\int |u(t)|^{p+1}. \tag{57}$$

Using (56), $\int |u(0)|^2 \geq A^2$ and $(\int |u(t)|^{p+1})^{2/(p+1)} \geq c_1(\Omega,p)\int |u(t)|^2$ (Hölder), we have the conclusion by integrating (57). ■

Now, the conclusion follows directly from lemma A.1 and the following lemma:

Lemma A.2 (Blow-up result for an integral inequality) *Let* $z \in \mathcal{C}([0,T), \mathbb{R}^+)$ *such that*

$$z(t) \geq B + a\int_0^t dt' z(t')^{(p+1)/2} - b\int_0^t dt' \int_0^{t'} ds z(t)^q \tag{58}$$

where $1 < p$, $1 < q < (p+1)/2$, $a > 0$ *and* $b > 0$.
There exists $B_0 > 0$ *such that if* $B \geq B_0$, *then* $T < +\infty$.

Proof:

Let $g(t) = \frac{a}{2}z(t)^{(p+1)/2} - b\int\int_0^t ds z(s)^q$. Let us show that $\forall t \in [0,T)$, $g(t) > 0$. We proceed by a priori estimates. For $B > 0$, we can define $T^* = \sup\{T' \in [0,T) | \forall t \in [0,T'), \int_0^t dt' g(t') \geq 0\} > 0$. Then we have $\forall t \in [0,T^*)$, $g(t) > 0$.

Indeed, we have $\forall t \in [0,T^*)$ $\int_0^t dt' g(t') \geq 0$. Therefore, (58) yields $z(t) \geq B + \frac{a}{2}\int_0^t dsz(s)^{(p+1)/2}$ which gives $z(t) \geq B$ and $z(t) \geq \frac{a}{2}\int_0^t dsz(s)^{(p+1)/2}$. Hence, $g(t) = \frac{a}{2}z(t)^{(p+1)/2} - b\int\int_0^t dsz(s)^q$
$\geq \frac{a}{2}B^{(p-1)/2}z(t) - b\int\int_0^t dsz(s)^q$
$> \frac{a}{2}B^{(p-1)/2}\frac{a}{2}\int_0^t dsz(s)^{(p+1)/2} - b\int\int_0^t dsz(s)^q$
$\geq \frac{a^2}{4}B^{(p-1)/2}\int_0^t dsB^{(p+1)/2-q}z(s)^q - b\int\int_0^t dsz(s)^q$
$= (\frac{a^2}{4}B^{p-q} - b)\int_0^t dsz(s)^q$. Now, if $B > (4ba^{-2})^{1/(p-q)}$, then $\forall t \in [0,T^*)$, $g(t) > 0$. This yields $T^* = T$ and $\forall t \in [0,T)$, $\int_0^t dt' g(t') \geq 0$.

Therefore, (58) implies that

$$\forall t \in [0,T), z(t) \geq B + \frac{a}{2}\int_0^t dsz(s)^{(p+1)/2}.$$

Hence, $T \leq \frac{4B^{(1-p)/2}}{a(p-1)} < +\infty$ by classical arguments. ■

B Appendix: Proof of lemma 3.3

Lemma 3.3 consists in a priori estimates on terms appearing in the integral equations satisfied by $\tilde{q}_1$ and $\tilde{q}_2$ (see (36), (37)). Let us recall them:

$$\begin{aligned}
\tilde{q}_1(s) &= K_1(s,\sigma)\tilde{q}_1(\sigma) + \int_\sigma^s d\tau K_1(s,\tau)V_{1,2}(\tau)\tilde{q}_2(\tau)\\
&+ \int_\sigma^s d\tau K_1(s,\tau)\tilde{B}_1(q)d\tau + \int_\sigma^s d\tau K_1(s,\tau)\tilde{R}_1^*(\tau)\\
&+ \int_\sigma^s d\tau K_1(s,\tau)\frac{d\theta}{ds}(\tau)\{\delta\tilde{\varphi}_1(\tau) + \tilde{\varphi}_2(\tau) + \delta\tilde{q}_1(\tau) + \tilde{q}_2(\tau)\}\\
\tilde{q}_2(s) &= K_2(s,\sigma)\tilde{q}_2(\sigma) + \int_\sigma^s d\tau K_2(s,\tau)V_{2,1}(\tau)\tilde{q}_1(\tau)\\
&+ \int_\sigma^s d\tau K_2(s,\tau)\tilde{B}_2(q)d\tau + \int_\sigma^s d\tau K_2(s,\tau)\tilde{R}_2^*(\tau)\\
&- \int_\sigma^s d\tau K_2(s,\tau)\frac{d\theta}{ds}(\tau)\{(1+\delta^2)\tilde{\varphi}_1(\tau) + \delta\tilde{\varphi}_2(\tau)) + (1+\delta^2)\tilde{q}_1(\tau)\\
&+ \delta\tilde{q}_2(\tau)\}
\end{aligned}$$

where K_1 is the fundamental solution of $\mathcal{L}+V_{1,1}$, K_2 is the fundamental solution of $\mathcal{L}-1+V_{2,2}$, $\mathcal{L}$ is given by (30),
$B(q) = (1+i\delta)\tilde{B}_1 + i\tilde{B}_2$,
$R^*(y,s) = (1+i\delta)\tilde{R}_1^* + i\tilde{R}_2^*$, B and R^* are given by (20).

From these expressions, we obviously see that the main step in doing a priori estimates is the understanding of the behavior of the kernels K_1 and K_2. By definition, K_1 and K_2 can be considered as perturbations of $e^{\theta\mathcal{L}}$ and $e^{\theta(\mathcal{L}-1)}$ respectively. Hence, we give the proof in two steps:

-in Step 1, we give estimates on the integral operators K_1 and K_2, nonlinear term $B(q)$ and corrective term R^* appearing in equations (36) and (37).

-in Step 2, we use these estimates to prove lemma 3.3.

Step 1: Estimates on linear, nonlinear and corrective terms of (36) and (37).

In order to estimate K_1 and K_2, we follow the perturbation method used in [18] (and before in Bricmont and Kupiainen [4]). Since K_1 and K_2 correspond respectively to the operators $\mathcal{L} + V_{1,1}$ and $\mathcal{L} - 1 + V_{2,2}$, we estimate first the potentials $V_{i,j}$ so we are able to adapt the cited method which compares K_1 and K_2 to $e^{\theta \mathcal{L}}$ and $e^{\theta(\mathcal{L}-1)}$ respectively. Then, we show that $B(q)$ can be considered in some sense as a quadratic term, and R^* is in fact small as $s \to +\infty$.

Lemma B.1 (Estimates on potentials $V_{i,j}$, $|\delta| \le 1/2$) *$\forall s \ge 1$,*

a)$V_{1,1}(y,s) \le Cs^{-1}$, $|\frac{d^n V_{1,1}}{dy^n}| \le Cs^{-n/2}$, $n = 0,1,2$, $|V_{1,1}(y,s)| \le Cs^{-1}(1+|y|^2)$, $V_{1,1}(y,s) = -\frac{1}{4s}h_2(y) + \tilde{V}_{1,1}(y,s)$ with $|\tilde{V}_{1,1}(y,s)| \le Cs^{-2}(1+|y|^4)$, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$, $\exists s_\epsilon$ such that

$$\sup_{s \ge s_\epsilon, \frac{|y|}{\sqrt{s}} \ge C_\epsilon} |V_{1,1}(y,s) - (-\frac{p-\delta^2}{p-1})| \le \epsilon$$

with $-\frac{p-\delta^2}{p-1} \le -1 - 1/(2p)$.

b)$V_{2,2}(y,s) \le Cs^{-1}$, $|\frac{d^n V_{2,2}}{dy^n}| \le Cs^{-n/2}$, $n = 0,1,2$, $|V_{2,2}(y,s)| \le Cs^{-1}(1+|y|^2)$, $V_{2,2}(y,s) = s^{-1}Q_\delta(y) + \tilde{V}_{2,2}(y,s)$ with Q_δ a polynomial of degree 2 with bounded coefficients and $|\tilde{V}_{2,2}(y,s)| \le Cs^{-2}(1+|y|^4)$, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$, $\exists s_\epsilon$ such that

$$\sup_{s \ge s_\epsilon, \frac{|y|}{\sqrt{s}} \ge C_\epsilon} |-1 + V_{2,2}(y,s) - (-1 - \frac{1+\delta^2}{p-1})| \le \epsilon$$

with $-1 - \frac{1+\delta^2}{p-1} < -1 - 1/p$.

c) For $V = V_{1,2}$ or $V_{2,1}$, we have $|V(y,s)| \le C|\delta|$, and $|V(y,s)| \le C|\delta| s^{-1}(1+|y|^2)$.

Proof:

The expressions of $V_{i,j}$ are given in lemma 3.2.

a) $V_{1,1}(y,s) \le (1-\delta^2)(|\varphi(0,s)|^{p-1} - \frac{1}{p-1}) + (p-1)|\varphi(0,s)|^{p-3}(|\varphi(0,s)|^2 - 0) - 1 \sim C(\delta)s^{-1} \le Cs^{-1}$.

We introduce $W_{1,1}(z,s) = V_{1,1}(y,s)$ with $z = y/\sqrt{s}$. In order to prove the next estimate, it is enough to prove that $|\frac{d^n W_{2,2}}{dy^n}| \le C$, $n = 0,1,2$. Since $V_{1,1}$ is a sum of products of terms $|\varphi|^{p-1}$ and $\varphi_j/|\varphi|$, $j = 1,2$, our problem reduces to proving that these terms have bounded first and second derivatives with respect to z, which follows easily (see (16), the key estimates are $\frac{\partial f_\delta}{\partial z} = \frac{-2bz}{(p-1)(p-1+bz^2)} f_\delta$ and $|f_\delta| \le |\varphi|$ with $b = \frac{(p-1)^2}{4(p-\delta^2)}$).

We introduce $\tilde{W}_{1,1}(Z,s) = V_{1,1}(y,s)$ with $Z = |y|^2/s$. We can Taylor expand $\tilde{W}_{1,1}$ near $Z = 0$ to have $\tilde{W}_{1,1}(Z,s) = \tilde{W}_{1,1}(0,s) + Z\frac{\partial \tilde{W}_{1,1}}{\partial Z}(0,s) + O(Z^2)$ with $\tilde{W}_{1,1}(0,s) = 1/(2s) + O(s^{-2})$ and $\frac{\partial \tilde{W}_{1,1}}{\partial Z}(0,s) = -1/4 + O(s^{-1})$. Returning to $V_{1,1}$, this yields the next two estimates.

The last estimate is obvious from the expressions of $V_{1,1}$ and φ.

b) For the first term, we make a change of variables by setting $Y = \kappa^{-1} f_\delta(y/\sqrt{s}) + 1/(2(p-\delta^2)s) \in (1/(2(p-\delta^2)s), 1/(2(p-\delta^2)s)+1]$ and $\hat{V}_{2,2}(Y,s) = V_{2,2}(y,s)$. Then, it is easy to see that $\hat{V}_{2,2}(.,s)$ is increasing. Therefore,
$\hat{V}_{2,2}(Y,s) \le \hat{V}_{2,2}(1/(2(p-\delta^2)s)+1, s) \sim C(\delta)s^{-1} \le Cs^{-1}$. For next estimates, do exactly as for $V_{1,1}$.

c) Same proofs, one has to be careful with the parameter δ. ■

Lemma B.2 (Estimates on K_1, $|\delta| < 1/2$) .
a) $\forall s \ge \tau > 1$ with $s \le 2\tau$, $\forall y, x \in \mathbb{R}$, $|K_1(s,\tau,y,x)| \le Ce^{(s-\tau)\mathcal{L}}(y,x)$, with
$e^{\theta\mathcal{L}}(y,x) = \frac{e^\theta}{\sqrt{4\pi(1-e^{-\theta})}} \exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}]$,
$\|K_1(s,\tau)(1-\chi(\tau))\|_{L^\infty} \le Ce^{-(s-\tau)/(2p)}$.

b) For each $A' > 0$, $A'' > 0$, $A''' > 0$, $\rho^* > 0$, there exists $s_9(A', A'', A''', \rho^*)$ with the following property:
$\forall s_0 \ge s_9$, assume that for $\sigma \ge s_0$,

$$\begin{aligned} |q_m(\sigma)| &\le A'\sigma^{-2}, m = 0, 1, |q_2(\sigma)| \le A''(\log\sigma)\sigma^{-2}, \\ |q_-(y,\sigma)| &\le A'''(1+|y|^3)\sigma^{-2}, \|q_e(\sigma)\|_{L^\infty} \le A''\sigma^{-\frac{1}{2}}, \end{aligned}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$\begin{aligned} |\alpha_2(s)| &\le A''\frac{\log\sigma}{s^2} + (s-\sigma)CA's^{-3}, \\ |\alpha_-(y,s)| &\le C(e^{-\frac{1}{2}(s-\sigma)}A''' + e^{-(s-\sigma)^2}A'')(1+|y|^3)s^{-2}, \\ \|\alpha_e(s)\|_{L^\infty} &\le C(A''e^{-\frac{(s-\sigma)}{p}} + A'''e^{(s-\sigma)})s^{-\frac{1}{2}}, \end{aligned}$$

where, as in decomposition (22),

$$K_1(s,\sigma)q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2} \alpha_m(s)h_m(y) + \alpha_-(y,s) + \alpha_e(y,s). \tag{59}$$

c) For each $A' > 0$, $A'' > 0$, $A''' > 0$, $\rho^* > 0$, there exists $s_{10}(A', A'', A''', \rho^*)$ with the following property:
$\forall s_0 \ge s_{10}$, assume that for $\sigma \ge s_0$,

$$\begin{aligned} |q_m(\sigma)| &\le A'\sigma^{-2}, m = 0, 1, |q_2(\sigma)| \le A''\sigma^{-3}, \\ |q_-(y,\sigma)| &\le A'''(1+|y|^3)\sigma^{-3}, \|q_e(\sigma)\|_{L^\infty} \le A'\sigma^{-2}, \end{aligned}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$\begin{aligned} |\alpha_2(s)| &\le A''s^{-3} + (s-\sigma)CA's^{-3}, \\ |\alpha_-(y,s)| &\le CA'''(1+|y|^3)s^{-3}, \end{aligned}$$

where $K_1(s,\sigma)q(\sigma)$ is expanded in (59).

Proof:

In [4] (proof of lemma 1), the authors prove the estimate for an integral operator K corresponding to $\mathcal{L}+V$ (see (30) for $\mathcal{L}$), where V is a particular function. However, their result is in fact true for a larger class of operators satisfying estimates of the type a) in lemma B.1. Hence, lemma B.2 follows. ■

Lemma B.3 (Estimates on K_2, $|\delta| < 1/2$) .
a) $\forall s \geq \tau > 1$ *with* $s \leq 2\tau$, $\forall y, x \in \mathbb{R}$,
$|K_2(s,\tau,y,x)| \leq Ce^{-(s-\tau)}e^{(s-\tau)\mathcal{L}}(y,x)$, *with*
$e^{\theta\mathcal{L}}(y,x) = \frac{e^\theta}{\sqrt{4\pi(1-e^{-\theta})}}\exp[-\frac{(ye^{-\theta/2}-x)^2}{4(1-e^{-\theta})}]$,
$\|K_2(s,\tau)(1-\chi(\tau))\|_{L^\infty} \leq Ce^{-(s-\tau)/p}$.

b) For each $A' > 0$, $A'' > 0$, $\rho^* > 0$, *there exists* $s_{11}(A', A'', \rho^*)$ *with the following property:*
$\forall s_0 \geq s_{11}$, *assume that for* $\sigma \geq s_0$,

$$\begin{aligned} |q_0(\sigma)| &\leq A'\sigma^{-2}, m=0,1, |q_\perp(y,\sigma)| \leq A'(1+|y|^3)\sigma^{-2}, \\ \|q_e(\sigma)\|_{L^\infty} &\leq A''\sigma^{-\frac{1}{2}}, \end{aligned}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$\begin{aligned} |\alpha_\perp(y,s)| &\leq C(e^{-\frac{1}{2}(s-\sigma)}A' + e^{-(s-\sigma)^2}A'')(1+|y|^3)s^{-2}, \\ \|\alpha_e(s)\|_{L^\infty} &\leq C(A''e^{-\frac{(s-\sigma)}{p}} + A')s^{-\frac{1}{2}}, \end{aligned}$$

where, as in decomposition (23),

$$K_2(s,\sigma)q(\sigma) = \alpha(y,s) = \alpha_0(s)h_0(y) + \alpha_\perp(y,s) + \alpha_e(y,s).$$

Proof:

Again, we can adapt the proof of lemma 1 in [4] with $\mathcal{L}$ replaced by $\mathcal{L}-1$ and V replaced by $V_{2,2}$, without difficulties. Indeed, one checks easily that $V_{2,2}$ satisfies all useful estimates: b) of lemma B.1. ■

Lemma B.4 (Estimates on $B(q(\tau))$ for $q(\tau)$ in $V_A(\tau)$, $|\delta| \leq 1/2$) .
$\forall A > 0$, $\exists s_{12}(A) > 0$ *such that* $\forall \tau \geq s_{12}(A)$, $q(\tau) \in V_A(\tau)$ *implies*
$|\chi(y,\tau)B(q(y,\tau))| = |(1+i\delta)\chi\tilde{B}_1 + i\chi\tilde{B}_2| \leq C|q|^2$,
$|B(q)| = |(1+i\delta)\tilde{B}_1 + i\tilde{B}_2| \leq C|q|^{\bar{p}}$ *with* $\bar{p} = \min(p,2)$.

Proof:

Start with (20) and do the same as in the proof of lemma 3.6 in [18]. ■

Lemma B.5 (Estimates on $R^*(y,s)$, $|\delta| \leq 1/2$) $\forall s \geq 1$, *if* R^* *is expanded as in (24), then:*
$|\tilde{R}^*_{1,0}(s)| \leq Cs^{-2}$, $\tilde{R}^*_{1,1}(s) = 0$, $|\tilde{R}^*_{1,2}(s)| \leq Cs^{-3}$,
$|\tilde{R}^*_{1,-}(y,s)| \leq Cs^{-2}(1+|y|^3)$, $\|\tilde{R}^*_{1,e}(s)\|_{L^\infty} \leq Cs^{-1}$, *and*
$|\tilde{R}^*_{2,0}(s)| \leq Cs^{-2}$, $|\tilde{R}^*_{2,\perp}(y,s)| \leq Cs^{-2}(1+|y|^3)$, $\|\tilde{R}^*_{2,e}(s)\|_{L^\infty} \leq Cs^{-1}$.

Proof: $\tilde{R}^*_{1,1}(s) = 0$ since R^* is even. All the other estimates follow from the three following estimates: $|\chi(y,s)R^*(y,s)| \le Cs^{-2}(1+|y|^2)$, $|R^*(y,s)| \le Cs^{-1}$ and $|\tilde{R}^*_{1,2}(s)| \le Cs^{-3}$.

Proof of $|\chi(y,s)R^*(y,s)| \le Cs^{-2}(1+|y|^2)$:
From (20), we have $R^*(y,s)$

$$\begin{aligned}
&= \frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - (1+i\delta)\frac{\varphi}{p-1} + (1+i\delta)|\varphi|^{p-1}\varphi \\
&= -(1+i\delta)\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}(-\frac{\kappa}{2(p-\delta^2)s^2} + \frac{(p-1)y^2}{4(p-\delta^2)s^2}f_\delta^p) \\
&+ (1+i\delta)\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}\frac{(p-1)y^2}{4(p-\delta^2)s}f_\delta^p \\
&+ (1+i\delta)i\delta\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta-1}(\frac{(p-1)y}{2(p-\delta^2)s}f_\delta^p)^2 \\
&+ (1+i\delta)\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}(-\frac{(p-1)}{2(p-\delta^2)s}f_\delta^p + \frac{p(p-1)^2y^2}{4(p-\delta^2)^2s^2}f_\delta^{2p-1}) \\
(60)&+ (1+i\delta)\kappa^{-i\delta}((f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{p+i\delta} - \frac{1}{p-1}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{1+i\delta}).
\end{aligned}$$

Some of these terms are easily seen to be bounded by $Cs^{-2}(1+|y|^2)$, whereas others need some calculation: we divide the others by $(1+i\delta)(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}\kappa^{-i\delta}$ and obtain $Q(y,s) = \frac{(p-1)y^2}{4(p-\delta^2)s}f_\delta^p - \frac{(p-1)}{2(p-\delta^2)s}f_\delta^p - \frac{1}{p-1}(f_\delta + \frac{\kappa}{2(p-\delta^2)s}) + (f_\delta + \frac{\kappa}{2(p-\delta^2)s})^p$. It remains to prove that $|\chi(y,s)Q(y,s)| \le Cs^{-2}(1+|y|^2)$. We write $Q(y,s) = (f_\delta + \frac{\kappa}{2(p-\delta^2)s})^p - f_\delta^p - \frac{(p-1)}{2(p-\delta^2)s}f_\delta^p - \frac{\kappa}{2(p-\delta^2)(p-1)s}$. Setting $z = \frac{|y|^2}{s} \ge 0$ and $\hat{Q}(z,s) = Q(y,s)$, we have $|\hat{Q}(0,s)| \le Cs^{-2}$ and $|\frac{\partial\hat{Q}}{\partial z}(z,s)| = p|\frac{\partial f_\delta}{\partial z}\{(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{p-1} - f_\delta^{p-1} - \frac{(p-1)}{2(p-\delta^2)s}f_\delta^{p-1}\}| \le Cs^{-1}$ if $z \le 2K_0$, (Taylor expansion). Therefore, if $z \le 2K_0$, $|\hat{Q}(z,s)| \le Cs^{-2} + O(|z|s^{-1})$. Returning to Q, this gives the result.

Proof of $|R^*(y,s)| \le Cs^{-1}$:
Thinking of R^* as a function of $|y|^2s^{-1}$ and s (see (60)), this estimate is obvious for all terms except $(1+i\delta)\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}(\frac{(p-1)|y|^2}{4(p-\delta^2)s}f_\delta^p + (f_\delta + \frac{\kappa}{2(p-\delta^2)s})^p - \frac{1}{p-1}f_\delta) = (1+i\delta)\kappa^{-i\delta}(f_\delta + \frac{\kappa}{2(p-\delta^2)s})^{i\delta}((f_\delta + \frac{\kappa}{2(p-\delta^2)s})^p - f_\delta^p)$. We conclude using a Taylor expansion.

Proof of $|\tilde{R}^*_{1,2}(s)| \le Cs^{-3}$:
From (60), we have

$$\tilde{R}^*_1(y,s) = -\frac{\partial\varphi_1}{\partial s} + \Delta\varphi_1 - \frac{1}{2}y.\nabla\varphi_1 + (|\varphi|^{p-1} - \frac{1}{p-1})(\varphi_1 - \delta\varphi_2).$$

Starting from $\tilde{R}^*_{1,2}(s) = \int d\mu(y)\chi(y,s)\tilde{R}^*_1(y,s)\frac{h_2(y)}{8}$, one carries out easy but long asymptotic calculation to get the result. ■

Step 2: Conclusion of the proof of lemma 3.3
We now prove lemma 3.3.

Ii) Case $\sigma \geq s_0$: Apply b) of lemma B.2 with $A' = A$, $A'' = A^2$ and $A''' = A$.

Case $\sigma = s_0$: From (25),
$\tilde{q}_1(y, s_0) = f_0(\frac{y}{\sqrt{s_0}})^p(d_0 + d_1 y/\sqrt{s_0}) - \Re((\frac{\kappa}{2(p-\delta^2)s_0})^{1+i\delta})$. Since (d_0, d_1) is chosen so that $(\tilde{q}_{1,0}(s_0), \tilde{q}_{1,1}(s_0)) \in \hat{V}_A(s_0)$, we have from lemma 3.1 in [18], $|\tilde{q}_{1,m}(s_0)| \leq A s_0^{-2}$, $m = 0, 1$, $|\tilde{q}_{1,2}(s_0)| \leq (\log s_0) s_0^{-2}$, $|\tilde{q}_{1,-}(y, s_0)| \leq C s_0^{-2}(1 + |y|^3)$ and $\|\tilde{q}_{1,e}(s_0)\|_{L^\infty} \leq s_0^{-1/2}$. We apply b) of lemma B.2 with $A' = A$, $A'' = C$, $A''' = 1$ to conclude

Iii): We have from lemma B.1 $|V_{1,2}(y, s)| \leq C|\delta| s^{-1}(1 + |y|^2)$.
Since $q(\tau) \in V_A(\tau)$, $|V_{1,2}(y, \tau)\tilde{q}_2(y, \tau)| \leq CA|\delta|\tau^{-3}(1 + |y|^4)$.
Hence, $|\iota_{1,2}(s)| = |C \int d\mu h_2(y) \int_\sigma^s d\tau K_1(s, \tau) V_{1,2}(\tau)\tilde{q}_2(\tau)|$
$\leq C \int d\mu(1 + |y|^2) \int_\sigma^s d\tau e^{(s-\tau)\mathcal{L}} CA|\delta|\tau^{-3}(1 + |x|^4)$
$\leq CA|\delta|\sigma^{-3} \int d\mu(1 + |y|^6)(s - \sigma)e^{s-\sigma}$
$\leq CA|\delta| s^{-3}(s - \sigma)e^{s-\sigma}$, if $\sigma \geq s_0 \geq \rho^*$.

If we set $Q(y, \tau) = V_{1,2}(y, \tau)\tilde{q}_2(y, \tau)$, we have by lemma B.1 $|V_{1,2}(y, \tau)| \leq C|\delta|$ and then $|Q_m(\tau)| \leq C|\delta| A\tau^{-2}$, $m = 0, 1, 2$, $|Q_-(y, \tau)| \leq C|\delta| A(1+|y|^3)\tau^{-2}$, $|Q_e(y, \tau)| \leq C|\delta| A^2 \tau^{-1/2}$. Applying lemma B.2 and integrating between σ and s yields good estimates for $\iota_{1,-}$ and $\iota_{1,e}$.

Iiii): Using lemma B.4 and a) of lemma B.2, we do the same as for the nonlinear term in Proof of lemma 3.4 in [18].

Iiv): From lemma B.5, we have $|\tilde{R}^*_{1,0}(\tau)| \leq C\tau^{-2}$, $\tilde{R}^*_{1,1}(\tau) = 0$, $|\tilde{R}^*_{1,2}(\tau)| \leq C\tau^{-3}$, $|\tilde{R}^*_{1,-}(y, \tau)| \leq C\tau^{-2}(1 + |y|^3)$, $|\tilde{R}^*_{1,e}(y, \tau)| \leq C\tau^{-1}$.
Applying lemma B.2 b) and integrating between σ and s gives the results for $\gamma_{1,2}$ and $\gamma_{1,-}$.
For $\gamma_{1,e}$,we use the following estimate: $|R^*(y, \tau)| \leq C\tau^{-1}$, and compute:
$|\gamma_{1,e}| = |\int_\sigma^s d\tau K_1(s, \tau) R_1^*(\tau)|$
$\leq \int_\sigma^s d\tau e^{(s-\sigma)\mathcal{L}} C\tau^{-1}$ (use lemma B.2 a))
$\leq C\sigma^{-1}(s - \sigma)e^{s-\sigma} \leq Cs^{-3/4}(s - \sigma)$ if $s_0 \geq s_5(\rho^*)$.

Iv): We set $Q(y, \tau) = \frac{d\theta}{ds}(\tau)\{\delta\tilde{q}_1 + \tilde{q}_2 + \delta\tilde{\varphi}_1 + \tilde{\varphi}_2\}$. By lemma 3.1, we have $|\frac{d\theta}{ds}(\tau)| \leq C\tau^{-2}$. Using $q(\tau) \in V_A(\tau)$, φ bounded and a simple calculation, we have:
$|Q_m(\tau)| \leq CA\tau^{-2}$, $m = 0, 1$, $|Q_2(\tau)| \leq C|\delta|\tau^{-3}$, $|Q_-(y, \tau)| \leq CA(1 + |y|^3)\tau^{-3}$, $|Q_e(y, \tau)| \leq C\tau^{-2}$.
Using lemma B.2 c), we obtain estimates for $\lambda_{1,2}$ and $\lambda_{1,-}$. For $\lambda_{1,e}$, use $|Q(y, \tau)| \leq C\tau^{-2}$ and do as for $\gamma_{1,e}$.

IIi): For $\sigma \geq s_0$, use lemma B.3.

For $\sigma = s_0$, we have from (25)
$\tilde{q}_2(y, s_0) = \frac{\alpha}{s_0}(\delta \cos[\delta \log(\frac{\alpha}{s_0})] - \sin[\delta \log(\frac{\alpha}{s_0})])(1 - \beta(s_0) f_0(\frac{y}{\sqrt{s_0}}))$ where α and $\beta(s_0)$ are given by (26). It follows easily that $\tilde{q}_{2,0}(s_0) = 0$, $|\tilde{q}_{2,\perp}(y, s_0)| \leq Cs_0^{-2}(1 + |y|^3)$ and $|\tilde{q}_{2,e}(y, s_0)| \leq Cs_0^{-1} \leq s_0^{1/2}$. Apply *b*) of lemma B.3 to conclude.

$IIii)$: we have by lemma B.1 $|V_{2,1}(y,\tau)| \leq C|\delta|$ and $|V_{2,1}(y,\tau)| \leq C|\delta|\tau^{-1}(1+|y|^2)$. If $Q(y,\tau) = V_{2,1}(y,\tau)\tilde{q}_1(y,\tau)$, then
$|Q_0(\tau)| \leq C|\delta|A^2 s^{-3}\log s$, $|Q_\perp(y,\tau)| \leq C|\delta|As^{-2}$ and
$|Q_e(y,\tau)| \leq C|\delta|A^2 s^{-1/2}$.
Using lemma B.3 b) yields the conclusion.

$IIiii)$: Using lemmas B.4 and lemma B.3 a), we do the same as for $Iiii)$.

$IIiv)$: Same estimates as $Iiv)$.

$IIv)$: By lemma 3.1, we have $|\frac{d\theta}{ds}(\tau)| \leq C\tau^{-2}$. Using lemma B.3 a) and integrating over $[\sigma, s]$ yields the conclusion.

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The author is supported by a grant from the M.E.S.R (Laboratoire d'analyse numérique, Paris VI).

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Chapitre 4

Reconnection of vortex with the boundary and finite time Quenching

Reconnection of vortex with the boundary and finite time Quenching†

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Abstract: We construct a stable solution of the problem of vortex reconnection with the boundary in a superconductor under the planar approximation. That is a solution of

$$\frac{\partial h}{\partial t} = \Delta h + e^{-h} H_0 - \frac{1}{h}$$

such that $h(0,t) \to 0$ as $t \to T$. We give a precise description of the vortex near the reconnection point and time.
We generalize the result to other quenching problems.
Mathematics subject Classification: 35K, 35B40, 35B45
Key words: quenching, blow-up, profile

1 Introduction

1.1 The physical motivation and results

We consider a Type II superconductor located in the region $z > 0$ of the physical space $\mathbb{R}^3$. Under some conditions, the magnetic field develops a particular type of line singularity called vortex (see Chapman, Hunton and Ockendon [5] for more details and discussion). In general, a vortex is not situated in a plane, but under some reasonable physical conditions, the planar approximation is relevant. In this case, a vortex line at time $t \geq 0$ can be viewed as $L(t) = \{(x,y,z) = (x,0,h(x,t)) | x \in \Omega\}$ where $\Omega = (-1,1)$ or $\Omega = \mathbb{R}$, and $h > 0$ is a regular function. The physical derivation gives that $h(x,t)$ satisfies the following equation:

$$h_t = h_{xx} + e^{-h} H_0 - F_0(h) \qquad \text{(I)}$$

where H_0 is the applied magnetic field assumed to be constant, F_0 is a regular function satisfying

$$\text{(1)} \qquad F_0(k) \sim \frac{1}{k} \text{ and } F_0'(k) \sim -\frac{1}{k^2} \text{ as } k \to 0.$$

We assume:
i) In the case where $\Omega = \mathbb{R}$

$$\text{(2)} \qquad \begin{cases} F_0(k) & \sim & Ce^{-2k} & as & k \to +\infty \\ |F_0'(k)| & \leq & Ce^{-2k} & as & k \to +\infty \\ h(x,t) & \sim & a_1 x + b_1 & as & x \to +\infty \\ h(x,t) & \sim & -a_2 x + b_2 & as & x \to -\infty \end{cases}$$

where $a_1 > 0$ and $a_2 > 0$. For simplicity, we take $b_1 = b_2 = 0$ and $a_1 = a_2$.

† Article paru dans Nonlinearity 10, 1997, pp. 1497-1550.

ii) In the case where $\Omega = (-1, 1)$,

$$h(1, t) = h(-1, t) = 1. \tag{3}$$

One can remark that boundary conditions of the type i) are closer to the physical context. Nevertheless, boundary conditions of the type *ii*) are mostly considered in the literature in order to simplify the mathematical approach of the problem.

Similar results can be shown with other types of boundary conditions (mixed boundary conditions on bounded domains). Indeed, our analysis will be local and therefore will not depend on boundary conditions.

Classical theory gives for any initial vortex line $L(0) = \{(x, 0, h_0(x)) | x \in \Omega\}$ where h_0 is positive, regular and satisfies boundary conditions, the existence and uniqueness of a solution to (I)-(2) and (I)-(3) locally in time. Therefore, there exists a unique solution to (I) on $[0, T)$ and either $T = +\infty$ or $T < +\infty$ and in this case $\lim_{t \to T} \inf_{x \in \Omega} h(x, t) = 0$, i.e. h extinguishes in finite time, and if $x_0 \in \Omega$ is such that there exists $(x_n, t_n) \to (x_0, T)$ as $n \to +\infty$ satisfying $h(x_n, t_n) \to 0$ as $n \to +\infty$, then x_0 is an extinction point of h.

This phenomenon is called a vortex reconnection with the boundary (the plane $z = 0$). Two questions arise:

- Question 1: Are there any initial data such that $T < +\infty$?
- Question 2: What does the vortex look like at the reconnection time?

Equation (I) with a more general exponent can also appear in various physical contexts (combustion for example), and the problem of reconnection is known as the quenching problem.

Indeed, we consider

$$h_t = \Delta h - F(h), \qquad h \geq 0 \tag{II}$$

where

$$\text{(H1)}\ F \in \mathcal{C}^\infty(\mathbb{R}_+^*), \quad F(k) \sim \frac{1}{k^\beta} \text{ and } F'(k) \sim -\frac{\beta}{k^{\beta+1}} \text{ as } k \to 0$$

with $\beta > 0$ and h is defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary condition $h \equiv 1$ on $\partial\Omega$. The case $\Omega = \mathbb{R}^N$ can also be considered with hypothesis (H1) and (H2) where

$$\text{(H2)} \begin{cases} |F(k)| + |F'(k)| & \leq & Ce^{-k} & as & k \to +\infty \\ h(x, t) & \sim & a_1|x| & as & |x| \to +\infty \end{cases}$$

Few results are known on equation (II). For $\beta > 0$, some criteria of quenching are known for solutions defined on $(-1, 1)$ with Dirichlet boundary conditions (or mixed boundary conditions) in dimension one (see Deng and Levine [6], Guo [12], Levine [18]). Even in that case, few informations are known on the solution at quenching except on the quenching rate (See also Keller and Lowengrub [17] for formal asymptotic behavior). In particular, there is no answer to questions 1 and 2 for problem (I).

To answer questions 1 and 2, we will not use the classical approach which consists in finding a general quenching criterion for initial data and in studying the quenching behavior of the solution. As in [22] and [25], the techniques we use here are the reverse: we study the quenching behavior of a solution a priori, and using this information, we prove by a priori estimates the existence of a solution which has all the properties we expect. Using this type of approach, we prove then that this behavior is stable. Let us first introduce:

$$\hat{\Phi}(z) = (\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2)^{1/(\beta+1)}, \tag{4}$$

and $H^*_{x_0}(x)$ defined by:
i) In the case $\Omega = \mathbb{R}^N$: $H^*_{x_0}(x) = H^*(x - x_0)$ where H^* is defined by:

$$\begin{array}{lll} H^*(x) = & \left[\frac{(\beta+1)^2|x|^2}{-8\beta\log|x|}\right]^{\frac{1}{\beta+1}} & \text{for } |x| \le C(a_1,\beta) \\ H^*(x) = & a_1|x| & \text{for } |x| \ge 1 \\ H^*(x) > & 0, |\nabla H^*(x)| > 0 & \text{for } x \neq 0 \text{ and } H^* \in \mathcal{C}^\infty(\mathbb{R}^N). \end{array} \tag{5}$$

ii) In the case where Ω is bounded:

$$\begin{array}{lll} H^*_{x_0}(x) = & \left[\frac{(\beta+1)^2|x-x_0|^2}{-8\beta\log|x-x_0|}\right]^{\frac{1}{\beta+1}} & \text{for } |x-x_0| \le \min\left(C(\beta), \frac{1}{4}d(x_0,\partial\Omega)\right) \\ H^*_{x_0}(x) = & 1 & \text{for } |x-x_0| \ge \frac{1}{2}d(x_0,\partial\Omega) \\ H^*_{x_0}(x) > & 0, |\nabla H^*(x)| > 0 & \text{for } x \neq x_0 \text{ and } H^*_{x_0} \in \mathcal{C}^\infty(\Omega\backslash\{x_0\}). \end{array}$$

We also introduce H, the set to initial data:

$$H = \{k \in \psi + H^1 \cap W^{2,\infty}(\mathbb{R}^N) \mid 1/k \in L^\infty(\mathbb{R}^N)\} \text{ if } \Omega = \mathbb{R}^N \tag{6}$$

where $\psi \in C^\infty(\mathbb{R}^N)$, $\psi \equiv 0$ for $|x| \le 1$, $\psi(x) = a_1|x|$ for $|x| \ge 2$ and a_1 is defined in (H2),

$$H = \{k \in H^1 \cap W^{2,\infty}(\Omega) \mid 1/k \in L^\infty(\Omega)\} \text{ if } \Omega \text{ is bounded.} \tag{7}$$

We claim the following:

Theorem (Existence and stability of a vortex reconnection with the boundary or quenching for equation (II) with $\beta > 0$)
Assume that $\Omega = \mathbb{R}^N$ and F is satisfying (H1) and (H2), or Ω is bounded and F is satisfying (H1).
1) **(Existence)** *For all $x_0 \in \Omega$, there exists a positive $h_0 \in H$ such that for a $T_0 > 0$, equation (II) with initial data h_0 has a unique solution $h(x,t)$ on $[0, T_0)$ satisfying $\lim_{t\to T_0} h(x_0,t) = 0$.*

Furthermore,
i)

$$\lim_{t\to T_0} \left\|\frac{(T_0-t)^{1/\beta+1}}{h(x_0 + z\sqrt{-(T_0-t)\log(T_0-t)}, t)} - \frac{1}{\hat{\Phi}(z)}\right\|_{L^\infty} = 0,$$

ii) $h^*(x) = \lim_{t\to T_0} h(x,t)$ exists for all $x \in \Omega$ and $h^*(x) \sim H^*_{x_0}(x)$ as $x \to x_0$.

2) **(Stability)** *For every $\epsilon > 0$, there exists a neighborhood $\mathcal{V}_0$ of h_0 in H with the following property:*

for each $\tilde{h}_0 \in \mathcal{V}_0$, there exist $\tilde{T}_0 > 0$ and $\tilde{x}_0$ satisfying

$$|T_0 - \tilde{T}_0| + |x_0 - \tilde{x}_0| \le \epsilon$$

such that equation (II) with initial data $\tilde{h}_0$ *has a unique solution* $\tilde{h}(x,t)$ *on* $[0,\tilde{T}_0)$ *satisfying* $\lim_{t\to\tilde{T}_0}\tilde{h}(t,\tilde{x}_0)=0$. *In addition,*

$$- \lim_{t\to\tilde{T}_0}\left\|\frac{(\tilde{T}_0-t)^{1/\beta+1}}{\tilde{h}(\tilde{x}_0+z\sqrt{-(\tilde{T}_0-t)\log(\tilde{T}_0-t)},t)}-\frac{1}{\hat{\Phi}(z)}\right\|_{L^\infty}=0,$$

- $\tilde{h}^*(x)=\lim_{t\to\tilde{T}_0}\tilde{h}(x,t)$ *exists for all* $x\in\Omega$ *and* $\tilde{h}^*(x)\sim H^*_{\tilde{x}_0}(x)$ *as* $x\to\tilde{x}_0$.

Remark: In the case $\beta=1$ (equation (I)), this Theorem implies that the vortex connects with the boundary in finite time. Let us note that the profile we obtain is $\mathcal{C}^1$ (which is not true for $\beta>1$). Using the precise estimate of the behavior of h at extinction, it will be interesting to check the validity of the planar approximation in the physical problem near the reconnection time for a behavior like the one described in the theorem.

Remark: We can also consider a larger class of equations:

$$\frac{\partial h}{\partial t}=\nabla.(A(x)\nabla h(x))-b(x)F(h)$$

where F satisfies (H1) and (H2) with $\beta>0$, $A(x)$ is a uniformly elliptic $N\times N$ matrix with bounded coefficients, $b(x)$ is bounded, and $b(x_0)>0$.
Using the stability result and techniques similar to [21], we can construct for arbitrary given k points in Ω a quenching solution h of equation (II) which quenches at time T exactly at the given points. The local quenching behavior of h near each of these points is the same as the one given in the Theorem.

Remark: We have two types of informations on the singularity:
- Part i): it describes the singularity in some refined scale variable at x_0 where we can observe the quenching dynamics. We point out that the estimate we obtain is global (convergence takes place in L^∞).
- Part ii): it describes the singularity in the original variables and shows its influence on the regular part of the solution.

We see in the estimates that these two descriptions are related.
In order to see why such a profile is selected, see [22] and [25] for similar discussions.

Remark: Part *ii*) is valid only for some extinction solutions. We suspect this kind of extinction behavior to be generic (see [15] for a related problem). Indeed, we suspect ourselves to be able to show existence of extinction solutions of (I)-(2) such that:

$$h(x,t)\to h^*_k(x)$$

where $h^*_k(x)\sim C|x|^k$, $k\in\mathbb{N}$ and $k\geq 2$. Unfortunately, this kind of behavior is suspected to be unstable.

1.2 Mathematical setting and strategy of the proof

The case $\Omega=\mathbb{R}^N$ is different from the case Ω is a bounded domain in the way how to treat the Cauchy problem outside the singularity.

Let us consider the problem of the existence of a solution such that *i*) and *ii*) of the Theorem hold. We first note that once the existence result is proved, the stability result can be proved in the same way as in [22]. In order to prove the Theorem, we use the following transformation:

$$u(x,t) = \frac{\alpha^{\frac{\alpha}{\beta+1}}}{h(x,t)^{\alpha}} \tag{8}$$

where h is the extinction solution of (II) to be constructed, and $\alpha > 0$. On its existence interval $[0, T)$, $u(t)$ satisfies

$$\frac{\partial u}{\partial t} = \Delta u - a\frac{|\nabla u|^2}{u} + f(u) \tag{III}$$

where $a = a(\alpha, \beta) = 1 + \frac{1}{\alpha}$,

$$f(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) = u^p + f_1(u) \tag{9}$$

with $p = p(\alpha, \beta) = \frac{1+\alpha+\beta}{\alpha}$,

$$\text{(H3)} \begin{cases} f_1 \in \mathcal{C}^{\infty}(\mathbb{R}_+), \ f_1(v) = o(v^p) \text{ and } f_1'(v) = o(v^{p-1}) \text{ as } v \to +\infty \\ 1 < a < p, \end{cases}$$

and in the case $\Omega = \mathbb{R}^N$,

$$\text{(H4)} \qquad \begin{cases} |f(v)| + |f'(v)| \leq C v^{1+\frac{1}{\alpha}} \exp(-\alpha^{\frac{1}{\beta+1}} v^{-\frac{1}{\alpha}}) \text{ as } v \to 0, \\ u(x,t) \sim \frac{1}{a_1 |x|} \text{ as } |x| \to +\infty \end{cases}$$

Now, with the transformation $(\alpha, \beta) \to (a(\alpha, \beta), p(\alpha, \beta))$, the problem of finding a solution h of (II) such that $\lim_{t\to T} \inf_{x\in\mathbb{R}} h(x,t) = 0$ is equivalent to the problem of finding a solution u of (III) such that

$$\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty,$$

(that is a solution of (III) which blows-up in finite time).

Problem (III) can be viewed as a gradient perturbation of the nonlinear heat equation $(a = 0)$

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1} u \tag{IV}$$

where $u(x,t)$ is defined for $x \in \mathbb{R}^N$, $t \geq 0$, $p > 1$ and $p < (N+2)/(N-2)$ if $N \geq 3$.

For this equation, Ball [1], Kavian [16] and Levine [20] obtained obstructions to global existence in time, using monotony properties and the maximum principle. Another method has been followed by Merle and Zaag in [22] (see also Giga and Kohn [10], [9] and [8], Bricmont and Kupiainen [4], Zaag [25]). Once an asymptotic profile (that is a function from which, after a time dependent scaling, $u(t)$ approaches as $t \to T$) is derived formally, the existence of a solution $u(t)$ which blows-up in finite time with the suggested profile is then proved rigorously, using analysis of equation (IV) near the given profile and reduction of the problem to a finite dimensional one.

In the case $a = 0$, the existence and stability of a blow-up solution $u(t)$ of (IV) such that at the blow-up point x_0:

$$\lim_{t\to T} \|(T-t)^{\frac{1}{p-1}} u(x_0 + \sqrt{(T-t)\log(T-t)}z, t) - \Phi_0(z)\|_{L^\infty} = 0$$

where

$$\Phi_0(z) = (p - 1 + \frac{(p-1)^2}{4p} z^2)^{-1/(p-1)}$$

is proved in [22]. Bricmont and Kupiainen obtained the existence result using renormalization group theory (see [4]).

In these new variables, and with the introduction of

$$\Phi(z) = (p - 1 + \frac{(p-1)^2}{4(p-a)}|z|^2)^{-\frac{1}{p-1}}. \tag{10}$$

(11) and $$U^*_{x_0}(x) = \alpha^{\frac{\alpha}{\beta+1}} H^*_{x_0}(x)^{-\alpha},$$

$= \left[\frac{8(p-a)|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}$ if $\Omega = \mathbb{R}^N$, $x_0 = 0$ and $|x| \le C(a_1, \beta)$, the Theorem is equivalent to the following Proposition:

Proposition 1 (Existence of blow-up solutions for equation (III)) *Assume that $\Omega = \mathbb{R}^N$ and f is satisfying (H3) and (H4), or Ω is bounded and f is satisfying (H3).*
For each $a \in (1, p)$, for each $x_0 \in \Omega$, there exist regular initial data u_0 such that equation (III) has a unique solution $u(x, t)$ which blows-up at a time $T_0 > 0$ only at the point x_0.

Moreover,
i) $\lim_{t\to T_0} u(x,t) = u^*(x)$ exists for all $x \in \Omega\backslash\{x_0\}$ and $u^*(x) \sim U^*_{x_0}(x)$ as $x \to x_0$.
ii)

$$\lim_{t\to T_0} \left\|(T_0 - t)^{\frac{1}{p-1}} u(x_0 + ((T_0 - t)|\log(T_0 - t)|)^{\frac{1}{2}} z, t) - \Phi(z)\right\|_{L^\infty} = 0.$$

Remark: This proposition provides us with a blow-up solution of (III) in the case $a \in (1, p)$. Let us remark that we already know that blow-up occurs in the case $a \le 1$:
- If $a < 1$ and $v = (1-a)^{\frac{1-a}{p-1}} u^{1-a}$, then v satisfies:

$$\frac{\partial v}{\partial t} = \Delta v + v^{p'} \quad \text{with} \quad p' = \frac{p-a}{1-a} > 1. \tag{12}$$

- If $a = 1$ and $v = (p-1)\log u$, then v satisfies

$$\frac{\partial v}{\partial t} = \Delta v + e^v. \tag{13}$$

It is well-known that equations (12) and (13) (and then (III)) have blow-up solutions.

We introduce *similarity variables* (see [10], [8] and [9])):

$$y = \frac{x - x_0}{\sqrt{T-t}}, s = -\log(T-t), w_{T,x_0}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{14}$$

where x_0 is the blow-up point and T the blow-up time of $u(t)$, a blow-up solution of (III) to be constructed (we will focus on the study of solutions that blow-up at one single point). We now assume $x_0 = 0$.
The study of the profile of u as $t \to T$ is then equivalent to the study of the asymptotic behavior of w_{T,x_0} (noted w) as $s \to \infty$, and each result for u has an equivalent formulation in terms of w. From equation (III), the equation satisfied by w is the following: $\forall y \in \mathbb{R}^N$, $\forall s \geq -\log T$:

$$(15) \qquad \frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} - a\frac{|\nabla w|^2}{w} + w^p + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}} w)$$

where $f_1(v) = f(v) - v^p$ and f satisfies (H3) and (H4).

The problem is then to find w a solution of (15) such that

$$\|w(y,s) - \Phi(\frac{y}{\sqrt{s}})\|_{L^\infty} \to 0 \text{ as } s \to +\infty.$$

We introduce

$$(16) \qquad \varphi(y,s) = \Phi(\frac{y}{\sqrt{s}}) + \frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s} \text{ and } q(y,s) = w(y,s) - \varphi(y,s)$$

where Φ is introduced in (10) (the introduction of the term $\frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s}$ is not necessary but it simplifies the calculations).

Then q satisfies: $\forall y \in \mathbb{R}^N$, $\forall s \geq -\log T$:

$$(17) \ \frac{\partial q}{\partial s} = (\mathcal{L} + V(y,s))q + B(q) + T(q) + R(y,s) + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi + q))$$

with $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}$,
$B(q) = (\varphi+q)^p - \varphi^p - p\varphi^{p-1}q$,
$T(q) = -a\frac{|\nabla\varphi + \nabla q|^2}{\varphi + q} + a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi}$.

Therefore, the question is to find w a solution of (15) or q a solution of (17) such that

$$(18) \qquad \lim_{s\to\infty} \|q(s)\|_{L^\infty} = 0.$$

The equation satisfied by q is almost the same as in [22], except the term $T(q)$. As in [22], we introduce estimates on q in the blow-up region $|z| \leq K_0$ or $|y| \leq K_0\sqrt{s}$, and in the regular region $|z| \geq K_0$ or $|y| \geq K_0\sqrt{s}$ where $z = \frac{y}{\sqrt{s}}$ is the self-similar variable for q. The estimates of $T(q)$ in the region $|y| \leq K_0\sqrt{s}$ follow from regularizing effect of the heat flow. One can remark that the Cauchy problem for an equation of the type $\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + u^p$ is suspected not to be solved in H^1 or $W^{1,p+1}$.

In the analysis of [22], the estimates in the region $|y| \geq K_0\sqrt{s}$ imply smallness of q only, and do not allow any control of $T(q)$ in this region. In other words, the analysis based on the method of [22], that is to estimate the solution in the z variable is not sufficient and must be improved. For this, we add estimates in three regions in a different variable scale (centered in the original x variable not necessarily at the considered blow-up point) using techniques similar to those used in [25] to derive the exact profile in x variable: $u(x,t) \to u^*(x)$ as

$t \to T$ where $u^*(x) \sim U^*(x)$ as $x \to 0$ (see (11) for U^*). This part makes the originality of the paper. We expect that such techniques can be useful in various supercritical problems.
We first define for $K_0 > 0$, $\epsilon_0 > 0$ and $t \in [0, T)$ given, three regions covering $\mathbb{R}^N$:

$$\begin{aligned}
P_1(t) &= \{x \mid |x| \le K_0\sqrt{-(T-t)\log(T-t)}\} \\
&= \{x \mid |y| \le K_0\sqrt{s}\} = \{x \mid |z| \le K_0\}, \\
P_2(t) &= \{x \mid \frac{K_0}{4}\sqrt{-(T-t)\log(T-t)} \le |x| \le \epsilon_0\} \\
&= \{x \mid \frac{K_0}{4}\sqrt{s} \le |y| \le \epsilon_0 e^{\frac{s}{2}}\} = \{x \mid \frac{K_0}{4} \le |z| \le \frac{e^{\frac{s}{2}}}{\sqrt{s}}\}, \\
P_3(t) &= \{x \mid |x| \ge \epsilon_0/4\} = \{x \mid |y| \ge \frac{\epsilon_0}{4}e^{\frac{s}{2}}\} = \{x \mid |z| \ge \frac{e^{\frac{s}{2}}}{\sqrt{s}}\},
\end{aligned}$$

$$\text{for } i = 1, 2, 3,\ P_i = \{(x,t) \in \mathbb{R}^N \times [0,T) | x \in P_i(t)\},$$

where $s = -\log(T-t)$, $y = \frac{x}{\sqrt{T-t}}$, $z = \frac{y}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\log(T-t)|}}$.

In P_1, the "extinction region" of h (which is also the blow-up region of u), we make the change of variables (14) and (16) to do an asymptotic analysis around the profile $\Phi(y/\sqrt{s})$.
Outside the singularity in region P_2, we control h using classical parabolic estimates on k, a rescaled function of h defined for $x \neq 0$ by

$$k(x,\xi,\tau) = (T-t(x))^{-\frac{1}{\beta+1}} h(x + \sqrt{T-t(x)}\xi, (T-t(x))\tau + t(x))$$

where $\frac{K_0}{4}\sqrt{(T-t(x))|\log(T-t(x))|} = |x|$. From equation (II), we see that k satisfies almost the same equation as h: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [-\frac{t(x)}{T-t(x)}, 1)$:

$$\frac{\partial k}{\partial \tau} = \Delta_\xi k - (T-t(x))^{\frac{\beta}{\beta+1}} F((T-t(x))^{\frac{1}{\beta+1}} k)$$

where $(T-t(x))^{\frac{\beta}{\beta+1}} F((T-t(x))^{\frac{1}{\beta+1}} k) \sim \frac{1}{k^\beta}$ as $(T-t(x))^{\frac{1}{\beta+1}} k \to 0$.
We will in fact prove that h behaves for $|\xi| \le \alpha_0\sqrt{|\log(T-t(x))|}$ and $\tau \in [\frac{t_0-t(x)}{T-t(x)}, 1)$ for some $t_0 < T$, like the solution of

$$\frac{\partial \hat{k}}{\partial \tau} = -\frac{1}{\hat{k}^\beta}.$$

In P_3, the regular region, we estimate directly h. This will give the desired estimate.

The proof of the existence result of the Theorem will be presented in section 2. Assuming some a priori estimates in P_1, P_2 and P_3, we show in section 2 that $h(t)$ can be controlled near the profile by a finite dimensional variable. Adjusting the finite dimensional parameters, we then conclude the proof. We present a priori estimates in P_1 in section 3, and in P_2 and P_3 in section 4.

The authors thank R. Kohn who pointed out various references on this problem. Part of this work was done while the second author was visiting the Institute for Advanced Study.

2 Existence of a blow-up solution for equation (16)

In this section, we give the proof of the existence result of the Theorem. The proof will be given in the case $\Omega = \mathbb{R}^N$ (we will mention the differences with the case Ω is bounded, when it is necessary, see section 4). We assume $N = 1$ in order to simplify the notations. The same calculations and proof hold in a higher dimension (see [22] and [25]). We assume $x_0 = 0$ since (II) is translation invariant. For simplicity in notations, we simplify hypothesis (H1) and assume that

$$\forall v \in (0,1],\ F(v) = \frac{1}{v^\beta}. \tag{19}$$

Same calculations holds without this simplification.

Let us first remark on the following about the Cauchy problem for equation (II).

Lemma 2.1 (Local Cauchy Problem for equation (II)) *The local in time Cauchy problem for equation (II) is well-posed in H where H is defined by (7) if Ω is bounded, and by (6) if $\Omega = \mathbb{R}$.*

Moreover, in both cases, either the solution h exists for all time $t > 0$ or only on $[0, T)$ with $T < +\infty$, and in this case $\lim_{t\to T} \inf_{x\in\Omega} h(x,t) = 0$.

Proof: The case Ω is bounded follows from classical arguments.

For the case $\Omega = \mathbb{R}$, we define $\tilde{h}(x,t)$ by $h(x,t) = \psi(x) + \tilde{h}(x,t)$. This way, (II) is equivalent to

$$\tilde{h}_t = \tilde{h}_{xx} - F(\psi(x) + \tilde{h}) + \psi_{xx}. \tag{20}$$

Using (H1) and (H2), we see by classical arguments that this equation can be solved in H. ■

Let us consider $\beta > 0$ and $T > 0$, all fixed. The problem is to find $t_0 < T$ and h_0 such that the solution of equation (II) with data at t_0 $h(x,t_0) = h_0$ extinguishes in finite time $T > 0$ at only one extinction point $x = 0$ and:

$$\text{- } \lim_{t\to T} \left\| \frac{(T-t)^{1/\beta+1}}{h(z\sqrt{-(T-t)\log(T-t)},t)} - \frac{1}{\hat{\Phi}(z)} \right\|_{L^\infty(\mathbb{R})} = 0 \tag{21}$$

- $h^*(x) = \lim_{t\to T} h(x,t)$ exists for all $x \in \mathbb{R}$ and

$$h^*(x) > 0 \text{ for } x \neq 0, h^*(x) \sim H^*(x) \text{ as } x \to 0 \tag{22}$$

where $\hat{\Phi}$ and H^* are introduced in (4) and (5).

As explained in the introduction, (21) and (22) follow from the control of $h(x,t)$ for $t \in [t_0, T)$ in three different scales, depending on the three regions P_1, P_2, and P_3.

a) In P_1, the extinction region, we rescale h by means of (8), (14) and (16) in order to define for $t \in [t_0, T)$, $q(s)$ where $s = -\log(T-t)$ and

$$\left\{ \begin{array}{llll} \forall y \in \mathbb{R}, & q(y,s) & = & (T-t)^{\frac{1}{p-1}} u(y\sqrt{T-t},t) - \varphi(y,s), \\ \forall x \in \mathbb{R}, & u(x,t) & = & \alpha^{\frac{\alpha}{\beta+1}} h(x,t)^{-\alpha} \text{ and } \alpha > 0, \\ & \varphi(y,s) & = & \Phi(\frac{y}{\sqrt{s}}) + \frac{(p-1)^{-\frac{1}{p-1}}}{2(p-a)s}, \\ p = \frac{\alpha+\beta+1}{\alpha}, & a = \frac{\alpha+1}{\alpha}, & & \text{and } \Phi \text{ is given in (10).} \end{array} \right. \tag{23}$$

Remark: To prove the Theorem, we can take $\alpha = 1$. Nevertheless, we need to keep $\alpha > 0$ general, if we want to deduce directly Proposition 1 from the Theorem.

The equation satisfied by q is (17): $\forall y \in \mathbb{R}$, $\forall s \geq -\log(T-t_0)$:

$$(24)\quad \frac{\partial q}{\partial s} = (\mathcal{L} + V(y,s))q + B(q) + T(q) + R(y,s) + e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi + q))$$

with $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}$,
$B(q) = (\varphi+q)^p - \varphi^p - p\varphi^{p-1}q$,
$T(q) = -a\frac{|\nabla\varphi + \nabla q|^2}{\varphi+q} + a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi}$,
$f_1(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - u^p$.

We note that $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbb{R}, d\mu)$ with

$$(25)\qquad d\mu(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}$$

and that its eigenvalues are $\{1 - \frac{m}{2} | m \in \mathbb{N}\}$.

In one dimension, $h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}$ is the eigenfunction corresponding to $1 - \frac{m}{2}$. We introduce also $k_m = h_m / \|h_m\|^2_{L^2(\mathbb{R},d\mu)}$ and note that Vect $\{h_m \mid m \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R}, d\mu)$.

We are interested in obtaining $L^\infty(\mathbb{R})$ estimates for q. Since $L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}, d\mu)$, we will expand q (actually, a cut-off of q) with respect to the eigenvalues of $\mathcal{L}$. Nevertheless, the estimates we will obtain will be L^∞ and not $L^2(\mathbb{R}, d\mu)$.

The control of $h(t)$ for $t \in [t_0, T)$ in this region P_1 is equivalent to the control of $q(s)$ for $s \in [-\log(T-t_0), +\infty)$ in a set $V_{K_0,A}(s)$ so that $\lim_{s\to\infty} \|q(s)\|_{L^\infty} = 0$. The definition of $V_{K_0,A}(s)$ requires the introduction of a cut-off function

$$(26)\qquad \chi(y,s) = \chi_0(\frac{|y|}{K_0\sqrt{s}})$$

where

$$(27)\qquad \chi_0 \in \mathcal{C}^\infty(\mathbb{R}^+, [0,1]),\ \chi_0 \equiv 1 \text{ on } [0,1],\ \chi_0 \equiv 0 \text{ on } [2, +\infty).$$

b) In P_2, we control a rescaled function of h defined for $x \neq 0$ by $\forall \xi \in \mathbb{R}$, $\forall \tau \in [\frac{t_0 - t(x)}{T - t(x)}, 1)$:

$$(28)\quad k(x,\xi,\tau) = (T - t(x))^{-\frac{1}{\beta+1}} h(x + \sqrt{T-t(x)}\xi, (T-t(x))\tau + t(x)),$$

where $t(x)$ is defined by

$$(29)\qquad |x| = \frac{K_0}{4}\sqrt{(T-t(x))|\log(T-t(x))|} = \frac{K_0}{4}\sqrt{\theta(x)|\log\theta(x)|}$$

$$\text{with} \qquad \theta(x) = T - t(x).$$

Let us note that $\theta(x)$ is related to the asymptotic profile $H^*(x)$.

Lemma 2.2 *For fixed K_0, we have:*
i) $H^*(x) \sim \hat{k}(1)\theta(x)^{\frac{1}{\beta+1}}$ *as* $x \to 0$,
ii) $|\nabla H^*(x)| \sim \frac{8}{(\beta+1)K_0}\frac{\hat{k}(1)}{\sqrt{|\log\theta(x)|}}\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}$ *as* $x \to 0$ *where*

$$\hat{k}(\tau) = ((\beta+1)(1-\tau) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16})^{\frac{1}{\beta+1}}. \tag{30}$$

Proof: From (29), we write:
$\log|x| = \log\frac{K_0}{4} + \frac{1}{2}\log\theta(x) + \frac{1}{2}\log|\log\theta(x)|$ and
$\frac{|x|^2}{-\log|x|} = \frac{2K_0^2}{16}\theta(x)\frac{\log\theta(x)}{\log\theta(x)+\log|\log\theta(x)|+2\log\frac{K_0}{4}}$. Therefore,

$$\log\theta(x) \sim 2\log|x| \text{ and } \theta(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\log|x||} \text{ as } x \to 0. \tag{31}$$

Since $H^*(x) = \hat{k}(1)\left[\frac{8|x|^2}{K_0^2|\log|x||}\right]^{\frac{1}{\beta+1}}$ and
$|\nabla H^*(x)| \sim \frac{4\sqrt{2}}{(\beta+1)K_0}\frac{\hat{k}(1)}{\sqrt{|\log|x||}}\left[\frac{8|x|^2}{K_0^2|\log|x||}\right]^{\frac{1}{\beta+1}-\frac{1}{2}}$ when x is small (see (5)), we get the conclusion. ■

k satisfies almost the same equation as h: $\forall\tau \in [\frac{t_0-t(x)}{\theta(x)}, 1)$, $\forall\xi \in \mathbb{R}$,

$$\frac{\partial k}{\partial\tau} = \Delta_\xi k - \theta(x)^{\frac{\beta}{\beta+1}}F(\theta(x)^{\frac{1}{\beta+1}}k). \tag{32}$$

We will see that the estimates on k allow us to write $\theta(x)^{\frac{\beta}{\beta+1}}F(\theta(x)^{\frac{1}{\beta+1}}k) = \frac{1}{k^\beta}$ for suitable ξ. If we show that $k(\tau)$ behaves like $\hat{k}$ (see (30)) which is a solution of the ODE

$$\frac{d\hat{k}}{d\tau} = -\frac{1}{\hat{k}^\beta}$$

defined for $\tau \in [0, \hat{T})$ with $\hat{T} = 1 + \frac{(\beta+1)K_0^2}{64\beta} > 1$, and that $|\nabla_\xi k(\tau)| \le \frac{C(K_0,A)}{\sqrt{|\log\theta(x)|}}$, then according to lemma 2.2, this yields that $h(x,t)$ behaves in P_2 like $H^*(x)$ and $|\nabla h(x,t)| \le C(K_0, A)|\nabla H^*(x)|$ if x and $T - t_0$ are small, which is almost the estimate *ii)* of the Theorem.

c) In P_3, we estimate directly h using the local in time well posedness of the Cauchy problem for equation (III).

More formally, we define for each $t \in [t_0, T)$ a set $S^*(t)$ depending on some parameters so that $h(t) \in S^*(t)$ means that h is controlled in the three regions as described before. We show then that if $\forall t \in [t_0, T)$, $h(t) \in S^*(t)$, then (21) and (22) hold and the Theorem follows.

Let us define $S^*(t)$:

Definition of $S^*(t)$ and S^*

I) For all $t_0 < T$, $K_0 > 0$, $\epsilon_0 > 0$, $\alpha_0 > 0$, $A > 0$, $\delta_0 > 0$, $C_0' > 0$, $C_0 > 0$ and $\eta_0 > 0$, for all $t \in [t_0, T)$, we define $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ as being the set of all functions $h \in H$ satisfying:

i) **Estimates in P_1**: *$q(s) \in V_{K_0,A}(s)$ where $s = -\log(T-t)$, $q(s)$ is defined in (23) and $V_{K_0,A}(s)$ is the set of all functions r in $W^{1,\infty}(\mathbb{R})$ such that*

$$\left\{\begin{array}{rclrcl} |r_m(s)| & \le & As^{-2}\ (m=0,1), & |r_2(s)| & \le & A^2 s^{-2}\log s, \\ |r_-(y,s)| & \le & As^{-2}(1+|y|^3), & |r_e(y,s)| & \le & A^2 s^{-1/2} \\ |(\frac{\partial r}{\partial s})_\perp(y,s)| & \le & As^{-2}(1+|y|^3), & & & \end{array}\right. \tag{33}$$

where

$$(34)\quad \begin{array}{rclrcl} r_e(y,s) & = & (1-\chi(y,s))r(y), & r_-(s) & = & P_-(\chi(s)r), \\ \textit{for } m\in\mathbb{N},\ r_m(s) & = & \int d\mu k_m(y)\chi(y,s)r(y), & r_\perp(s) & = & P_\perp(\chi(s)r), \end{array}$$

χ is defined in (26), P_- and $P_\perp$ are the $L^2(\mathbb{R},d\mu)$ projectors respectively on Vect $\{h_m|m\geq 3\}$ and Vect $\{h_m|m\geq 2\}$, $d\mu$, h_m and k_m are introduced in (25).

ii) **Estimates in** P_2*: For all $|x|\in[\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|},\epsilon_0]$, $\tau=\tau(x,t)=\frac{t-t(x)}{\theta(x)}$, and $|\xi|\leq\alpha_0\sqrt{|\log\theta(x)|}$, $|k(x,\xi,\tau)-\hat{k}(\tau)|\leq\delta_0$, $|\nabla_\xi k(x,\xi,\tau)|\leq\frac{C_0'}{\sqrt{|\log\theta(x)|}}$, and $|\nabla_\xi^2 k(x,\xi,\tau)|\leq C_0$ where k, $\hat{k}$, $t(x)$ and $\theta(x)$ are defined in (28), (30) and (29).*

iii) **Estimates in** P_3*: For all $|x|\geq\frac{\epsilon_0}{4}$, $|h(x,t)-h(x,t_0)|\leq\eta_0$ and $|\nabla h(x,t)-\nabla h(x,t_0)|\leq\eta_0$.*

II) For all $t_0<T$ we define $S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)=$ $\{k\in C([t_0,T),H)\mid\forall t\in[t_0,T),k(t)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)\}$.

Remark: Note that according to (25) and (34), we have for all $r\in L^\infty(\mathbb{R})$,

$$(35)\qquad r(y) = \sum_{m=0}^{2} r_m(s)h_m(y)+r_-(y,s)+r_e(y,s),$$

$$(36)\qquad r(y) = \sum_{m=0}^{1} r_m(s)h_m(y)+r_\perp(y,s)+r_e(y,s).$$

Therefore, $i)$ yields an estimate on $\|q(s)\|_{L^\infty}$ and $\|\left(\frac{\partial q}{\partial y}\right)_\perp(s)\|_{L^\infty}$.
Remark: The estimates on h are in $W^{1,\infty}(\mathbb{R})$. In particular, they are global. The estimates on $\frac{\partial q}{\partial y}$ in P_1, $\nabla_\xi k$ in P_2 and on ∇h in P_3 allow us to control the term $T(q)$ appearing in the equation satisfied by q (see (24)). We remark that the estimate $q(s)\in V_{K_0,A}(s)$ describes h mainly in P_1. The estimate on q_e involved in definition (33) is useful only in the frontier between P_1 and P_2.

Now we show that if we find suitable parameters and initial data such that $h\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)$, then the Theorem holds.

Proposition 2.1 (Reduction of the proof) *For given $t_0<T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 and η_0 such that $\delta_0\leq\frac{1}{2}\hat{k}(1)$ and $\eta_0\leq\frac{1}{2}\inf_{|x|\geq\epsilon_0/4}h(x,t_0)$, assume that $h\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0)$. Then $h(t)$ extinguishes in finite time T only at the point $x_0=0$, that is $\lim_{t\to T}h(0,t)=0$ and $\forall x\neq 0$, there exists $\eta(x)>0$ such that*

$$(37)\qquad \liminf_{t\to T}\ \min_{|x'-x|\leq\eta(x)} h(x',t)>0.$$

Moreover, with $\hat{\Phi}$ and H^* defined by (4) and (5),

$$(38)\qquad \lim_{t\to T}\|\frac{(T-t)^{\frac{1}{\beta+1}}}{h(z\sqrt{-(T-t)\log(T-t)},t)}-\frac{1}{\hat{\Phi}(z)}\|_{L^\infty(\mathbb{R})}=0,$$

$h^*(x) = \lim_{t\to T} h(x,t)$ *exists for all* $x \in \mathbb{R}$ *and*

$$h^*(x) > 0 \textit{ for } x \neq 0 \textit{ and } h^*(x) \sim H^*(x) \textit{ as } x \to 0. \tag{39}$$

Proof: We assume that $h \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. One can remark that once (38), (37) and (39) are proved, it follows that
i) $\lim_{t\to T} h(0,t) = 0$: $h(t)$ extinguishes at time T at the point $x = 0$,
ii) $x = 0$ is the only extinction point of h.
It remains then to prove (37), (38) and (39).

Proof of (37):
From $iii)$ of Definition of $S^*(t)$, we know that if $|x| \geq \frac{\epsilon_0}{4}$, then $\forall t \in [t_0, T)$, $h(x,t) \geq h(x,t_0) - \eta_0 \geq \inf_{|x|\geq\frac{\epsilon_0}{4}} h(x,t_0) - \eta_0 \geq \frac{1}{2}\inf_{|x|\geq\frac{\epsilon_0}{4}} h(x,t_0)) > 0$. This yields (37) for $|x| \geq \epsilon_0$.

From $ii)$ of Definition of $S^*(t)$, we have $\forall |x| \in (0, \epsilon_0]$, for t close enough to T, $|k(x,0,\tau(x,t)) - \hat{h}(\tau(x,t))| \leq \delta_0$ where $\tau(x,t) = \frac{t-t(x)}{\theta(x)}$. Therefore, $k(x,0,\tau(x,t)) \geq \hat{k}(\tau(x,t)) - \delta_0 \geq \hat{k}(1) - \delta_0 \geq \frac{1}{2}\hat{k}(1)$ (from (30) and $\delta_0 \leq \frac{1}{2}\hat{k}(1)$). From (28), it follows: $h(x,t) \geq \frac{1}{2}\hat{k}(1)\theta(x)^{\frac{1}{\beta+1}} > 0$. This yields (37) for $0 < |x| < \epsilon_0$.

Proof of (38):
We consider $q(s)$, the function introduced in (23). Let us show that

$$\|q(s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to +\infty. \tag{40}$$

From $i)$ of the definition of $S^*(t)$ and (35), we have $\forall s \in [-\log(T-t_0), +\infty)$, $q(s) \in V_{K_0,A}(s)$ and
$|q(y,s)| = |1_{\{|y|\leq 2K_0\sqrt{s}\}}\left(\sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y,s)\right) + q_e(y,s)|$
$\leq 1_{\{|y|\leq 2K_0\sqrt{s}\}}(As^{-2}(1+|y|) + A^2 s^{-2}\log s(|y|^2+2) + As^{-2}(1+|y|^3)) + A^2 s^{-1/2} \leq C(K_0, A)s^{-1/2}$ and (40) follows.

Let $z \in \mathbb{R}$ and $g(z) = |(T-t)^{1/\beta+1}/h(z\sqrt{-(T-t)\log(T-t)}, t) - \frac{1}{\hat{\Phi}(z)}|$. We have
$g(z)$
$\leq C|(T-t)^{\frac{\alpha}{\beta+1}}\alpha^{\frac{\alpha}{\beta+1}} h(z\sqrt{-(T-t)\log(T-t)}, t)^{-\alpha} - \alpha^{\frac{\alpha}{\beta+1}}\hat{\Phi}(z)^{-\alpha}|^{\frac{1}{\bar{\alpha}}}$
where $\bar{\alpha} = \max(\alpha, 1)$.

Using (4) and (23), we have $\alpha = 1/(p-a)$ and $\beta = (p-a)/(a-1)$, therefore $\frac{\alpha}{\beta+1} = \frac{1}{p-1}$,

$$\alpha^{\frac{\alpha}{\beta+1}}\hat{\Phi}(z)^{-\alpha} = (\frac{\beta+1}{\alpha} + \frac{(\beta+1)^2}{4\beta\alpha}|z|^2)^{-\frac{1}{p-1}} = \varphi(z\sqrt{s}, s) - \frac{(p-1)^{-1/(p-1)}}{2(p-a)s},$$

and $(T-t)^{\frac{\alpha}{\beta+1}}\alpha^{\frac{\alpha}{\beta+1}} h(z\sqrt{-(T-t)\log(T-t)}, t)^{-\alpha}$
$= (T-t)^{\frac{1}{p-1}} u(z\sqrt{-(T-t)\log(T-t)}, t)$ with $s = -\log(T-t)$.

Combining this with (23) again, we get
$g(z) \leq C(\alpha,\beta)\left(|q(z\sqrt{-\log(T-t)}, -\log(T-t))| + 1/|\log(T-t)|\right)^{\frac{1}{\bar{\alpha}}}$
$\leq C\left(\|q(s)\|_{L^\infty(\mathbb{R})} + 1/|\log(T-t)|\right)^{\frac{1}{\bar{\alpha}}} \to 0$ as $t \to T$ by (40). This yields (38).

Proof of (39): From the proof of (37) and classical theory (see Merle [21] for a similar problem), there exists a profile function $h^*(x)$ such that $\forall x \neq 0$, $\lim_{t\to T} h(x,t) = h^*(x) > 0$. To show that $h^*(x) \sim H^*(x)$ as $x \to 0$, we give the following localization estimate:

Proposition 2.2 (Localization in P_2) *Assume that k is a solution of equation*

$$k_\tau = \Delta k - \frac{1}{k^\beta} \tag{41}$$

for $\tau \in [0, \tau_0)$ with $\tau_0 \leq 1 (< \hat{T})$. Assume in addition: $\forall \tau \in [0, \tau_0]$,
i) For $|\xi| \leq 2\xi_0$, $|k(\xi, 0) - \hat{k}(0)| \leq \delta$ and $|\nabla k(\xi, 0)| \leq \delta$,
ii) For $|\xi| \leq \frac{7\xi_0}{4}$, $k(\xi,\tau) \geq \frac{1}{2}\hat{k}(\tau)$.
iii) For $|\xi| \leq \frac{7\xi_0}{4}$, $|\nabla^2 k(\xi,\tau)| \leq C_0$,
where $\hat{k}$ is introduced in (30). Then there exists $\epsilon = \epsilon(\delta, \xi_0)$ such that $\forall \tau \in [0,\tau_0]$, for $|\xi| \leq \xi_0$,
$|k(\xi,\tau) - \hat{k}(\tau)| \leq \epsilon$ and $|\nabla k(\xi,\tau)| \leq \epsilon$, where $\epsilon \to 0$ as $\delta \to 0$ and $\xi_0 \to +\infty$.

Proof: We prove in section 4 a more accurate version of this Proposition (Proposition 4.1). One can adapt without difficulties the proof to the present context. ■

Let us apply this Proposition to $k(x,\xi,\tau)$ when x is near zero with $\tau_0 = 1$ and $\xi_0 = |\log\theta(x)|^{1/4}$. We first check all the hypothesizes of the Proposition:

Lemma 2.3 *If x is small enough, then $k(x,\xi,\tau)$ satisfies (41) for $|\xi| \leq |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Moreover,*

$$(42) i) \quad \sup_{|\xi| \leq |\log\theta(x)|^{1/4}} |k(x,\xi,0) - \hat{k}(0)| + |\nabla_\xi k(x,\xi,0)| \leq \delta(x) \to 0 \text{ as } x \to 0,$$

ii) for $|\xi| \leq |\log\theta(x)|^{\frac{1}{4}}$, $\forall \tau \in [0,1)$, $k(x,\xi,\tau) \geq \frac{1}{2}\hat{k}(\tau)$,
iii) for $|\xi| \leq |\log\theta(x)|^{\frac{1}{4}}$, $\forall \tau \in [0,1)$, $|\nabla_\xi^2 k(x,\xi,\tau)| \leq C_0$.

Combining this lemma and Proposition 2.2, we get $\forall \tau \in [0,1)$, $|k(x,\xi,\tau) - \hat{k}(\tau)| \leq \epsilon(x) \to 0$ as $x \to 0$. Using (28), (30) and letting $\tau \to 1$, we obtain

$$\theta(x)^{-\frac{1}{\beta+1}} h^*(x) \sim \hat{k}(1) = \left(\frac{(\beta+1)^2 K_0^2}{64\beta}\right)^{\frac{1}{\beta+1}}. \tag{43}$$

By lemma 2.2, we obtain $h^*(x) \sim H^*(x)$ as $x \to 0$, which concludes the proof of Proposition 2.1.

Proof of lemma 2.3:

$i)$ and $iii)$: Since (29) implies that $\theta(x) \to 0$ as $x \to 0$, we have by combining (38) and (28):

$\sup_{|\xi| \leq |\log\theta(x)|^{1/4}} |1/k(x,\xi,0) - 1/\hat{\Phi}(\frac{x + \xi\sqrt{\theta(x)}}{\sqrt{\theta(x)|\log\theta(x)|}})| \to 0$ as $x \to 0$. Hence, from (4), the first part of (42) follows.

From $ii)$ of the Definition of $S^*(t)$, we have $|\nabla_\xi k(x,\xi,0)| \leq \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ and $|\nabla_\xi^2 k(x,\xi,0)| \leq C_0$ for $|\xi| \leq |\log\theta(x)|^{1/4}$, if x is small. This yields the second part of $i)$ and $iii)$.

ii): From *ii)* of the Definition of $S^*(t)$, it follows that for x small enough, we have $|k(x,\xi,\tau) - \hat{k}(\tau)| \le \delta_0$ for $|\xi| \le |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Hence, *ii)* follows from (30) since $\delta_0 \le \frac{1}{2}\hat{k}(1)$. By the way, this implies that $|\theta(x)^{\frac{1}{\beta+1}} k(x,\xi,\tau)| \le 1$ for $|\xi| \le |\log\theta(x)|^{1/4}$ and $\tau \in [0,1)$. Therefore, it follows from (32) and (19) that k satisfies (41). ■

From this Proposition, the proof of the Theorem reduces to find suitable parameters $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0, η_0 and $h_0 \in H$ so that the solution h of equation (II) with data $h(t_0) = h_0$ belongs to
$S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$.

Unfortunately, the spectrum of $\mathcal{L}$ which greatly determines the dynamic of q (and then the dynamic of h too) contains two expanding eigenvalues: 1 and 1/2. Therefore, we expect that for most choices of initial data h_0, the corresponding $q_0(s)$ and $q_1(s)$ with $s = -\log(T-t)$ will force $h(t)$ to exit $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$.

As a matter of fact, we will show through a priori estimates that for suitably chosen $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 and η_0, the control of $h(t)$ in $S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for $t \in [t_0, T)$ reduces to the control of $(q_0(s), q_1(s))$ in

(44) $$\hat{V}_A(s) \equiv [-As^{-2}, As^{-2}]^2$$

for $s \ge -\log(T-t_0)$ ($q_0(s)$ and $q_1(s)$ correspond to expanding eigenvalues in the q variable). Hence, we will consider initial data h_0 depending on two parameters $(d_0, d_1) \in \mathbb{R}^2$, and then, we will fix (d_0, d_0) using a topological argument so that $(q_0(s), q_1(s)) \in \hat{V}_A(s)$ for all $s \ge -\log(T-t)$, which yields $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$, thanks to the finite dimensional reduction.

Let us define

$$h_0(d_0, d_1, x) = (T-t_0)^{\frac{1}{\beta+1}} \alpha^{\frac{1}{\beta+1}} \left\{ \Phi(z) + (d_0 + d_1 z)\chi_0\left(\frac{|z|}{K_0/16}\right) \right\}^{-\frac{1}{\alpha}} \chi_1(x, t_0)$$

(45) $$+ H^*(x)(1 - \chi_1(x, t_0))$$

where $z = x/\sqrt{(T-t_0)|\log(T-t_0)|}$,

(46) $$\chi_1(x, t_0) = \chi_0\left(\frac{x}{(T-t_0)^{\frac{1}{2}}|\log(T-t_0)|^{\frac{p}{2}}}\right),$$

Φ, χ_0 and H^* are defined in (10), (27) and (5). The problem now reduces to find (d_0, d_1) in some $\mathcal{D} \subset \mathbb{R}^2$ such that
$h(d_0, d_1) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$.

The proof is divided in two parts:

i) Finite dimensional reduction:

From the technique of a priori estimates, we find suitable parameters $t_0 < T$, K_0, ϵ_0,α_0, A, δ_0, C_0', C_0 and η_0 so that the following property is true: Assume that for $t_* \in [t_0, T)$, we have $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ and
$h(t_*) \in \partial S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_*)$, then

$(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$ where $s_* = -\log(T-t_*)$, q_0 and q_1 follow from q by (34), q and $\hat{V}_A(s)$ are defined in (23) and (44).

ii) Solution of the finite dimensional problem:

We use a topological argument to find a parameter $(d_0, d_1) \in \mathbb{R}^2$ such that $(q_0(s), q_1(s)) \in \hat{V}_A(s)$ for all $s \geq -\log(T-t_0)$, and therefore, $h \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. This yields the Theorem.

Part I: A priori estimates of $h(t)$, solution of equation (II) and finite dimensional reduction

Step 0: Initialization of the problem

We claim the following lemma:

Lemma 2.4 (Initialization of the problem) *There exists $K_{01} > 0$ such that for each $K_0 \geq K_{01}$ and $\delta_1 > 0$, $\exists \alpha_1(K_0, \delta_1) > 0$ and $C^*(K_0) > 0$ such that $\forall \alpha_0 \leq \alpha_1(K_0, \delta_1)$, $\exists \epsilon_1(K_0, \delta_1, \alpha_0) > 0$, such that $\forall \epsilon_0 \leq \epsilon_1(K_0, \delta_1, \alpha_0)$, $\forall C_1 > 0$, $\forall A \geq 1$, $\exists t_1(K_0, \delta_1, \epsilon_0, A, C_1) < T$ such that $\forall t_0 \in [t_1, T)$, there exists a rectangle $\mathcal{D}(t_0, K_0, A) \subset \mathbb{R}^2$ with the following properties:*
If $h(x, t_0)$ is defined by (45), then:

i) $\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$, $h(t_0) \in H$ defined in (6), $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$ defined in (44) and $h(t_0) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_1, C^*(K_0), C_1, 0, t_0)$, with $s_0 = -\log(T-t_0)$. More precisely:

$$\begin{array}{llll} |q_0(s_0)| & \leq A s_0^{-2} & |q_1(s_0)| & \leq A s_0^{-2} \\ |q_2(s_0)| & \leq s_0^{-2}\log s_0 & |q_-(y, s_0)| & \leq C s_0^{-2}(1+|y|^3) \\ |q_e(y, s_0)| & \leq s_0^{-1/2} & |\left(\frac{\partial q}{\partial y}\right)_\perp (y, s_0)| & \leq s_0^{-2}(1+|y|^3), \\ |\frac{\partial q}{\partial y}(y, s_0)| & \leq s_0^{-\frac{1}{2}} & \text{for } |y| \geq K_0\sqrt{s_0}, & \end{array}$$

for all $|x| \in [\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|}, \epsilon_0]$, $\tau_0 = \frac{t_0 - t(x)}{\theta(x)}$, and $|\xi| \leq 2\alpha_0\sqrt{|\log \theta(x)|}$, $|k(x, \xi, \tau_0) - \hat{k}(\tau_0)| \leq \delta_1$, $|\nabla_\xi k(x, \xi, \tau_0)| \leq \frac{C^*(K_0)}{\sqrt{|\log \theta(x)|}}$ and $|\nabla_\xi^2 k(x, \xi, \tau_0)| \leq C_1$ where k, $\hat{k}$, $t(x)$ and $\theta(x)$ are defined in (28), (30) and (29).

ii) $$(d_0, d_1) \in \mathcal{D}(t_0, K_0, A) \Leftrightarrow (q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0),$$

$$(d_0, d_1) \in \partial\mathcal{D}(t_0, K_0, A) \Leftrightarrow (q_0(s_0), q_1(s_0)) \in \partial\hat{V}_A(s_0),$$

$(q_0(s_0), q_1(s_0))$ is an affine function of (d_0, d_1) when $(d_0, d_1) \in \partial\mathcal{D}(t_0, K_0, A)$.

Proof: See Appendix A.

Step 1: A priori estimates

We now claim the following estimates:

Proposition 2.3 (A priori estimates in P_1) *There exists $K_{02} > 0$ such that for each $K_0 \geq K_{02}$, there exists $A_2(K_0) > 0$ such that for each $A \geq A_2(K_0)$, $\epsilon_0 > 0$ and $C_0' \leq A^3$, there exist $\eta_2(\epsilon_0) > 0$ and $t_2(K_0, \epsilon_0, A, C_0') < T$ such that for each $t_0 \in [t_2(K_0, \epsilon_0, A, C_0'), T)$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$ and $\eta_0 \leq \eta_2(\epsilon_0)$, we have the following property:*

- if $h(x, t_0)$ is given by (45) and if (d_0, d_1) is chosen so that $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$ defined in (44) with $s_0 = -\log(T-t_0)$,

- *if for some* $t_* \in [t_0, T)$, *we have*
$\forall t \in [t_0, t_*]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ *then*

$$\begin{array}{rclrcl} |q_2(s_*)| & \leq & A^2 s_*^{-2}\log s_* - s_*^{-3}, & |q_-(y,s_*)| & \leq & \frac{A}{2} s_*^{-2}(1+|y|^3) \\ |q_e(y,s_*)| & \leq & \frac{A^2}{2} s_*^{-1/2}, & |(\frac{\partial q}{\partial y})_\perp(y,s_*)| & \leq & \frac{A}{2} s_*^{-2}(1+|y|^3), \end{array}$$

where $s_* = -\log(T - t_*)$, q *is defined in (23) and the notation is given in (34).*

Proof: See section 3. ■

Proposition 2.4 (A priori estimates in P_2**)** *There exists* $K_{03} > 0$ *such that for all* $K_0 \geq K_{03}$, $\delta_1 \leq 1$, $\xi_0 \geq 1$, $C_0^* > 0$, $C_0'^* > 0$ *and* $C_0''^* > 0$ *we have the following property:*
Assume that k *is a solution of equation*

$$\frac{\partial k}{\partial \tau} = \Delta k - \frac{1}{k^\beta} \tag{47}$$

for $\tau \in [\tau_1, \tau_2)$ *with* $0 \leq \tau_1 \leq \tau_2 \leq 1$ $(< \hat{T})$.
Assume in addition: $\forall \tau \in [\tau_1, \tau_2]$,
i) $\forall \xi \in [-2\xi_0, 2\xi_0]$, $|k(\xi, \tau_1) - \hat{k}(\tau_1)| \leq \delta_1$ *and* $|\nabla k(\xi, \tau_1)| \leq \frac{C_0''^*}{\xi_0}$,
ii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $|\nabla k(\xi, \tau)| \leq \frac{C_0'^*}{\xi_0}$ *and* $|\nabla^2 k(\xi, \tau)| \leq C_0^*$,
iii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $k(\xi, \tau) \geq \frac{1}{2}\hat{k}(\tau)$,
where $\hat{k}$ *is given by (30). Then, for* $\xi_0 \geq \xi_{03}(C_0'^*, C_0^*, C_0''^*)$ *there exists* $\epsilon = \epsilon(K_0, C_0'^*, \delta_1, \xi_0)$ *such that* $\forall \xi \in [-\xi_0, \xi_0]$, $\forall \tau \in [\tau_1, \tau_2]$,
$|k(\xi, \tau) - \hat{k}(\tau)| \leq \epsilon$ *and* $|\nabla k(\xi, \tau)| \leq \frac{2C_0''^*}{\xi_0}$, *where* $\epsilon \to 0$ *as* $(\delta_1, \xi_0) \to (0, +\infty)$.

Proof: See section 4. ■

Proposition 2.5 (A priori estimates in P_3**)** *For all* $\epsilon > 0$, $\epsilon_0 > 0$, $\sigma_0 > 0$, *and* $\sigma_1 > 0$, *there exists* $t_4(\epsilon, \epsilon_0, \sigma_0, \sigma_1) < T$ *such that* $\forall t \in [t_4, T)$, *if* h *is a solution of (II) on* $[t_0, t_*]$ *for some* $t_* \in [t_0, T)$ *satisfying*
i) for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$, $\forall t \in [t_0, t_*]$,

$$\sigma_0 \leq h(x,t) \leq \sigma_1,\ |\nabla h(x,t)| \leq \sigma_1 \text{ and } |\nabla^2 h(x,t)| \leq \sigma_1, \tag{48}$$

ii) $h(x, t_0) = H^*(x)$ *for* $|x| \geq \frac{\epsilon_0}{6}$ *where* H^* *is defined by (5),*
then for $|x| \in [\frac{\epsilon_0}{4}, +\infty)$, $\forall t \in [t_0, t_*]$,

$$|h(x,t) - h(x,t_0)| + |\nabla h(x,t) - \nabla h(x,t_0)| \leq \epsilon.$$

Proof: See section 4. ■

Step 2: Finite dimensional reduction
From Propositions 2.3, 2.4 and 2.5, we have the following:

Proposition 2.6 (Finite dimensional reduction) *We can choose parameters* $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0' *and* C_0 *and* η_0 *such that the following properties hold: Assume that* $h(x, t_0)$ *is given by (45) and* $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$. *Then,*
i) $h(t_0) \in H \cap S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_0)$.

Assume in addition that for some $t_* \in [t_0, T)$, *we have* $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ *and*
$h(t_*) \in \partial S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_*)$ *then*
ii) $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$ *where q is defined in (23) and* $s_* = -\log(T - t_*)$.
iii) **(Transversality)** *there exists* $\nu_0 > 0$ *such that* $\forall \nu \in (0, \nu_0)$,
$(q_0(s_* + \nu), q_1(s_* + \nu)) \notin \hat{V}_A(s_* + \nu)$ *(hence*
$h(t_* + \nu) \notin S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_* + \nu)$*).*

Proof: We proceed in two steps: we first show that we can fix K_0, δ_0 and C_0 independently from A, take $A \geq A_7$ and choose ϵ_0, α_0, C_0', η_0 and t_0 in terms of A, so that $i)$ and $ii)$ hold. In the second step, we fix A and t_0 so that $iii)$ holds too.

Proof of $i)$ *and* $ii)$
It follows from the following lemma:

Lemma 2.5 *There exist constants* K_0, δ_0, C_0, *and* $A_7 > 0$ *such that for all* $A \geq A_7$, *there exist* $\epsilon_0(A) > 0$, $\alpha_0(A)$, $C_0'(A)$, $\eta_7(A)$ *and* $t_7(A) < T$ *such that for all* $\eta_0 \leq \eta_7$ *and* $t_0 \in [t_7, T)$, *and under the hypotheses of Proposition 2.6,* $i)$ *and* $ii)$ *hold.*

Proof
Let us first choose suitably the constants, and then show that $i)$ and $ii)$ of Proposition 2.6 follow for this choice.

All the constants we are referring to below appear either in lemma 2.4 or Propositions 2.3, 2.4 or 2.5.

We proceed in ten steps:
i) Fix $K_0 = 4\max(K_{01}, K_{02}, K_{03})$.
ii) Fix $\delta_0 = \frac{1}{4}\min(\hat{k}(1), 1)$ (note that $\hat{k}(1)$ depends only on K_0). Fix $C_0 = 1$. Let $A_7(K_0)$ be large enough so that $A_7 \geq \max(1, A_2(K_0))$ and for all $A \geq A_7(K_0)$, $A^3 \geq C_0'(A)$ where we introduce
$C_0'(A) = 4\max\left(C_3 A^2 K_0^3 + \|\nabla\hat{\Phi}\|_{L^\infty(B(0,2K_0))}, \frac{20\hat{k}(1)}{(\beta+1)K_0}, C^*(K_0)\right)$ with
$C^*(K_0)$ defined in lemma 2.4 and C_3 a constant which is independent of all the parameters and appears in lemma 2.6.
Consider A any number larger than $A_7(K_0)$, and consider $C_0'(A)$.
iii) Applying Proposition 2.4 with K_0, $C_0^* = 2$, $C_0'^*(A) = 2C_0'(A)$ and $C_0''^*(A) = \frac{1}{4}C_0'(A)$, we get $\xi_0^*(A) \geq 1$ and $\delta_1^*(A) \leq 1$ such that for all $\xi_0 \geq \xi_0^*$ and $\delta_1 \leq \delta_1^*$, the conclusion of the Proposition holds with $\epsilon = \frac{\delta_0}{2}$.
iv) Let $\delta_1(A) = \min(\frac{1}{2}\delta_1^*(A), \delta_0)$ and $C_1 = \frac{1}{2}$.
v) We claim the following lemma:

Lemma 2.6 $\forall A \geq A_7$, *there exist* $\alpha_5(K_0, \delta_1(A)) > 0$ *such that for all* $\alpha_0 \leq \alpha_5$, *there exists* $\epsilon_5(\alpha_0, A) > 0$ *such that for all* $\epsilon_0 \leq \epsilon_5(\alpha_0, A)$, *there are* $t_5(\epsilon_0, A) < T$ *and* $\eta_5(\epsilon_0, A) > 0$ *such that for all* $\eta_0 \leq \eta_5(\epsilon_0, A)$ *and* $t_0 \in [t_5(\epsilon_0, A), T)$,
if for all $t \in [t_0, t_*]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ *for some* $t_* \in [t_0, T)$, *then we have for* $|x| \in [\frac{K_0}{4}\sqrt{(T - t_*)|\log(T - t_*)|}, \epsilon_0]$:
i) For $|\xi| \leq \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$ *and for all* $\tau \in [\max(0, \frac{t_0 - t(x)}{\theta(x)}), \frac{t_* - t(x)}{\theta(x)}]$:
$k(x, ., .)$ *satisfies (47) and,* $|\nabla_\xi k(x, \xi, \tau)| \leq \frac{2C_0'(A)}{\sqrt{|\log\theta(x)|}}$, $|\nabla_\xi^2 k(x, \xi, \tau)| \leq 2C_0$ *and* $k(x, \xi, \tau) \geq \frac{1}{2}\hat{k}(\tau)$.

ii) For $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$ *and* $\tau = \max(\frac{t_0-t(x)}{\theta(x)}, 0)$*:* $|k(x,\xi,\tau) - \hat{k}(\tau)| \le \delta_1$ *and* $|\nabla_\xi k(x,\xi,\tau)| \le \frac{C_0'(A)}{4\sqrt{|\log\theta(x)|}}$.

Proof: We focus on the proof of the fact that for $|x| \in (0,\epsilon_0]$,
for $|\xi| \le \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$, for $t \in [\max(0,t(x)),T)$, we have

$$|\nabla_\xi k(x,\xi,\tau)| \le \frac{2C_0'(A)}{\sqrt{|\log\theta(x)|}} \tag{49}$$

where $\tau = \frac{t-t(x)}{\theta(x)}$, and: for $|x| \in (0,\epsilon_0]$, for $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$|k(x,\xi,\tau_0(x)) - \hat{k}(\tau_0(x))| \le \delta_1 \tag{50}$$

$$\text{and } |\nabla_\xi k(x,\xi,\tau_0(x))| \le \frac{\frac{1}{4}C_0'(A)}{\sqrt{|\log\theta(x)|}} \tag{51}$$

where $\tau_0(x) = \max(\frac{t_0-t(x)}{\theta(x)}, 0)$.
The other estimates follow by similar techniques.

Let $\delta > 0$ to be fixed later. If $\alpha_0 \le \alpha_7(K_0,\delta)$ for some $\alpha_7(K_0,\delta) > 0$, then we have from (29): for $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$(1-\delta)|x| \le |x + \xi\sqrt{\theta(x)}| \le (1+\delta)|x|. \tag{52}$$

Proof of (49):
From (28), we have

$$\nabla_\xi k(x,\xi,\tau) = \theta(x)^{-\frac{1}{\beta+1}+\frac{1}{2}}\nabla h(x+\xi\sqrt{\theta(x)}, t). \tag{53}$$

Let us denote $x + \xi\sqrt{\theta(x)}$ by X and distinguish three cases:

- Case where $|X| \le \frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|}$:
From (8), we write $\nabla h(X,t) = C\frac{\nabla u}{u^{1+\frac{1}{\alpha}}}(X,t)$.
From $i)$ of the Definition of $S^*(t)$, we get
$|(T-t)^{\frac{1}{p-1}}u(X,t) - \Phi(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})|$
$= |q(\frac{X}{\sqrt{T-t}}, -\log(T-t)) + \frac{\kappa}{2(p-a)|\log(T-t)|}| \le \frac{CA^2K_0^3}{\sqrt{|\log(T-t)|}}$ by lemma B.1. Moreover,
$|\nabla u(X,t) - (T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}\nabla\Phi(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})| =$
$(T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\nabla q(\frac{X}{\sqrt{T-t}}, -\log(T-t))|$
$\le (T-t)^{-\frac{1}{p-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}CA^2K_0^3$ (see the proof of lemma B.1)
Hence, by (9), we obtain:
$|(T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}}\nabla h(X,t) - |\log(T-t)|^{-\frac{1}{2}}\nabla\hat{\Phi}(\frac{X}{\sqrt{(T-t)|\log(T-t)|}})|$
$\le \frac{C_3A^2K_0^3}{\sqrt{|\log(T-t)|}}$ and
$|\nabla h(X,t)| \le \left(C_3A^2K_0^3 + \|\nabla\hat{\Phi}\|_{L^\infty(B(0,K_0))}\right)(T-t)^{\frac{1}{\beta-1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}$.
This gives by (53):
$|\nabla_\xi k(x,\xi,\tau)| \le \left(\frac{T-t}{\theta(x)}\right)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log(T-t)|^{-\frac{1}{2}}C_0'(A)$.

Since $(1-\delta)|x| \leq |X|$ (see (52)) and $|X| \leq K_0\sqrt{(T-t)|\log(T-t)|}$, we have $|x| \leq \frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\log(T-t)|}$.
From (29), we have $|x| \to \theta(x)$ is an increasing function. Therefore,
$\theta(x) \leq \theta(\frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\log(T-t)|}) \sim \frac{8}{K_0^2}\frac{K_0^2(T-t)|\log(T-t)|}{16(1-\delta)^2\frac{1}{2}|\log(T-t)|} = \frac{(T-t)}{(1-\delta)^2}$ by (31).
Moreover, we have $t \geq t(x)$, therefore, $T-t \leq \theta(x)$. Hence,
$|\nabla_\xi k(x,\xi,\tau)| \leq 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$ if δ is small enough.

- Case where $|X| \in [\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)}, \epsilon_0]$:
We write $\nabla h(X,t) = \theta(X)^{\frac{1}{\beta+1}-\frac{1}{2}}\nabla_\xi k(X,0,\frac{t-t(X)}{\theta(X)})$. This gives by (53):
$\nabla_\xi k(x,\xi,t) = \left(\frac{\theta(X)}{\theta(x)}\right)^{\frac{1}{\beta+1}-\frac{1}{2}}\nabla_\xi k(X,0,\frac{t-t(X)}{\theta(X)})$.
From $ii)$ of the Definition of $S^*(t)$, we obtain:
$|\nabla_\xi k(x,\xi,\tau)| \leq C_0'(A)|\log\theta(x)|^{-\frac{1}{2}} \times \frac{\theta(X)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(X)|^{-\frac{1}{2}}}{\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(x)|^{-\frac{1}{2}}}$.
Using (52) and taking δ small enough, this yields
$|\nabla_\xi k(x,\xi,\tau)| \leq 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$.

- Case $|X| \geq \epsilon_0$: If $\eta_0 \leq \delta \min_{|x'|\geq\epsilon_0}|\nabla h(x',t_0)|$, then we have from $iii)$ of the Definition of $S^*(t)$:
$|\nabla h(X,t)| \leq (1+\delta)|\nabla h(X,t_0)| \leq (1+\delta)|\nabla h(\gamma x,t_0)|$ where $\gamma = 1-\delta$ if $\beta > 1$ and $\gamma = 1+\delta$ if $\beta \leq 1$ (see (52)).
From lemma 2.2, we get:
$|\nabla h(X,t)| \leq (1+\delta)\frac{10\hat{k}(1)}{(\beta+1)K_0}\theta(\gamma x)^{\frac{1}{\beta+1}-\frac{1}{2}}|\log\theta(\gamma x)|^{-\frac{1}{2}}$.
Arguing as before, we obtain from (53):
$|\nabla_\xi k(x,\xi,\tau)| \leq \frac{20\hat{k}(1)}{(\beta+1)K_0}|\log\theta(x)|^{-\frac{1}{2}} \leq 2C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$ if δ is small enough. This concludes the proof of (49).

Proof of (50):
If $|x| \geq \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then (29) yields $t(x) \leq t_0$ and $\tau_0(x) = \frac{t_0-t(x)}{\theta(x)}$. Hence, (50) follows from lemma 2.4.
If $|x| \leq \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then $t(x) \geq t_0$ and $\tau_0(x) = 0$. From (28) and (30), we let $X = x+\xi\sqrt{\theta(x)}$ and write:
$|k(x,\xi,0)-\hat{k}(0)| = |\theta(x)^{-\frac{1}{\beta+1}}h(X,t(x)) - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}| \leq I + II$
where $I = |\theta(x)^{-\frac{1}{\beta+1}}h(X,t(x)) - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{|X|^2}{\theta(X)|\log\theta(x)|}\right)^{\frac{1}{\beta+1}}|$
and $II = |\left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{|X|^2}{\theta(X)|\log\theta(x)|}\right)^{\frac{1}{\beta+1}} - \left((\beta+1)+\frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}|$.
From $i)$ of the Definition of $S^*(t)$, (23) and the fact that
$|X| \leq (1+\delta)|x| \leq \frac{(1+\delta)K_0}{4}\sqrt{\theta(x)|\log\theta(x)|} \leq K_0\sqrt{\theta(x)|\log\theta(x)|}$, we get
$I \leq CA^2K_0^3|\log\theta(x)|^{-\frac{1}{2}} \leq CA^2K_0^3|\log(T-t_0)|^{-\frac{1}{2}}$, since
$|x| \leq \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$. Now, if $T-t_0$ is small enough, then $I \leq \frac{\delta_1}{2}$.
From (52) and (29), we have $(1-\delta)^2\frac{K_0^2}{16} \leq \frac{|X|^2}{\theta(X)|\log\theta(X)|} \leq (1+\delta)^2\frac{K_0^2}{16}$. Hence, if δ is small enough, we obtain $II \leq \frac{\delta_1}{2}$.
This concludes the proof of (50).

Proof of (51):
If $|x| \geq \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then (29) yields $t(x) \leq t_0$ and $\tau_0(x) =$

$\frac{t_0-t(x)}{\theta(x)}$. Hence, lemma 2.4 yields: for $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$,
$|\nabla_\xi k(x,\xi,\tau_0(x))| \le C^*(K_0)|\log\theta(x)|^{-\frac{1}{2}} \le \frac{1}{4}C_0'(A)$.
If $|x| \le \frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)|}$, then $t(x) \ge t_0$ and $\tau_0(x) = 0$. With $X = x + \xi\sqrt{\theta(x)}$, we write: $\nabla_\xi k(x,\xi,0) = \theta(x)^{-\frac{1}{\beta+1}+\frac{1}{2}}\nabla h(X,t(x))$. Arguing as for the first case in the proof of (49), we get:
$|\nabla_\xi k(x,\xi,0)| \le \left[C_3A^2K_0^3 + \|\nabla\hat{\Phi}\|_{L^\infty(B(0,K_0))}\right]|\log\theta(x)|^{-\frac{1}{2}}$
$\le \frac{1}{4}C_0'(A)|\log\theta(x)|^{-\frac{1}{2}}$.

This concludes the proof of (51) and the proof of lemma 2.6. ■

vi) We now fix $\alpha_0(A) = \min(\frac{1}{2}\alpha_1(K_0,\delta_1(A)),\alpha_5(K_0,\delta_1(A)),1)$. We also fix $\epsilon_0(A) \le \min(\epsilon_1(K_0,\delta_1(A),\alpha_0(A)),\epsilon_5(\alpha_0(A),A))$ such that
$\alpha_0(A)\sqrt{|\log\theta(\epsilon_0)|} \ge \xi_0^*(A)$.
vii) Then, we take $\eta_7(A) = \frac{1}{2}\min(\eta_2(\epsilon_0(A)),\eta_5(\epsilon_0(A),A))$ and consider $\eta_0 \le \eta_7$.
viii) By direct parabolic estimates, it is easy to see that there exists $t_6(A) < T$ such that for all $t_0 \in [t_6,T)$, if
$h(t_0) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_1,\eta_0,t_0)$ and $\forall t \in [t_0,t']$,
$h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$, then
$h(t') \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',\frac{3}{4},\eta_0,t')$.
ix) Let $\sigma_0(A) = \frac{1}{2}\hat{k}(1)\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}}$ and $\sigma_1(A) = \max(\frac{3}{2}\hat{k}(0)\theta(\frac{\epsilon_0}{4})^{\frac{1}{\beta+1}}$,
$C_0'\frac{\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(\frac{\epsilon_0}{6})|}}, C_0'\frac{\theta(\frac{\epsilon_0}{4})^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(\frac{\epsilon_0}{4})|}}, C_0\theta(\frac{\epsilon_0}{6})^{\frac{1}{\beta+1}-1})$.
x) Let $t_7(A) = \max(t_1(K_0,\delta_1(A),\epsilon_0(A),A,C_1),t_2(K_0,\epsilon_0(A),A,C_0'(A))$,
$t_4(\frac{\eta_0}{2},\epsilon_0,\sigma_0,\sigma_1),t_5(\epsilon_0(A),A),t_6(A))$, and consider t_0 an arbitrary number in $[t_7(A),T)$.

Now, we show that $i)$ and $ii)$ of Proposition 2.6 hold for this choice. Let us assume that $h(t_0)$ is given by (45) and $(d_0,d_1) \in \mathcal{D}(t_0,K_0,A)$. Then, lemma 2.4 applies and $h(t_0) \in H \cap S^*(t_0.K_0,\epsilon_0.\alpha_0,A,\delta_1,C^*(K_0),0,t_0)$. Since $\delta_1 \le \delta_0$, $C^*(K_0) \le C_0'$ and $0 < \eta_0$, $i)$ follows.
We now assume that in addition, we have $\forall t \in [t_0,t_*]$,
$h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ and
$h(t_*) \in \partial S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t_*)$ for some $t_* \in [t_0,T)$. According to the Definition of $S^*(t)$, three cases may occur:

Case 1: $q(s_*) \in \partial V_{K_0,A}(s_*)$. From $ii)$ of lemma 2.4, Proposition 2.3 and $i)$ of the Definition of $S^*(t)$, we have $(q_0(s_*),q_1(s_*)) \in \partial\hat{V}_A(s_*)$ which is $i)$ of Proposition 2.6.

Case 2: There exist x and ξ such that
$|x| \in [\frac{K_0}{4}\sqrt{(T-t_*)|\log(T-t_*)|},\epsilon_0]$ and $|\xi| \le \alpha_0\sqrt{|\log\theta(x)|}$, and either
$|k(x,\xi,\tau_1) - \hat{k}(\tau_1)| = \delta_0$ or $|\nabla_\xi k(x,\xi,\tau_1)| = \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ or $|\nabla_\xi^2 k(x,\xi,\tau_1)| = C_0 = 1$, where $\tau_1 = \frac{t_*-t(x)}{\theta(x)} < 1$.

According to viii) and lemma 2.4, we have $|\nabla_\xi^2 k(x,\xi,\tau_*)| \le \frac{3}{4}$. Let $\tau_0 = \max(\frac{t_0-t(x)}{\theta(x)},0)$ and $\xi_0 = \alpha_0\sqrt{|\log\theta(x)|}$. Note that $\xi_0 \ge \alpha_0\sqrt{|\log\theta(\epsilon_0)|} \ge \xi_0^*$.
Since $\alpha_0 \le 1$, it follows from lemma 2.6:
- For $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, $|k(x,\xi,\tau_0) - \hat{k}(\tau_0)| \le \delta_1$ and $|\nabla_\xi k(x,\xi,\tau_0)| \le \frac{C_0'(A)}{4\sqrt{|\log\theta(x)|}} \le \frac{C_0'(A)}{4\xi_0}$.

- For $|\xi| \le \frac{7}{4}\alpha_0\sqrt{|\log\theta(x)|}$ and for all $\tau \in [\tau_0, \tau_1]$: $k(x,.,.)$ satisfies (87) and $|\nabla_\xi k(x,\xi,\tau)| \le \frac{2C_0'(A)}{\xi_0}$, $|\nabla_\xi^2 k(x,\xi,\tau)| \le 2C_0$ and $k(x,\xi,\tau) \ge \frac{1}{2}\hat{k}(\tau)$.
Applying Proposition 2.4 yields:
For $|\xi| \le \alpha_0\sqrt{|\log\theta(x)|}$, $|k(x,\xi,\tau_1)-\hat{k}(\tau_1)| \le \frac{\delta_0}{2}$ and $|\nabla_\xi k(x,\xi,\tau_1)| \le \frac{2\frac{1}{4}C_0'(A)}{\sqrt{|\log\theta(x)|}}$ $< \frac{C_0'(A)}{\sqrt{|\log\theta(x)|}}$, which contradicts the hypotheses of Case 2.

Case 3: There exists $x \in \mathbb{R}$ such that $|x| \ge \frac{\epsilon_0}{4}$ and $|h(x,t_*) - h(x,t_0)| = \eta_0$ or $|\nabla h(x,t_*) - \nabla h(x,t_0)| = \eta_0$. From *ii*) of the Definition of $S(t)$, we have: $\forall t \in [t_0, t_*]$, for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$: $|k(x,0,\tau) - \hat{k}(\tau)| \le \delta_0$, $|\nabla_\xi k(x,0,\tau)| \le \frac{C_0'}{\sqrt{|\log\theta(x)|}}$ and $|\nabla_\xi^2 k(x,0,\tau)| \le C_0$, where $\tau = \frac{t-t(x)}{\theta(x)}$. Using (28) and the fact that $\delta_0 \le \frac{1}{2}\hat{k}(1) \le \frac{1}{2}\hat{k}(0)$, we obtain:
$\frac{1}{2}\hat{k}(1)\theta(x)^{\frac{1}{\beta+1}} \le h(x,t) \le \frac{3}{2}\hat{k}(0)\theta(x)^{\frac{1}{\beta+1}}$, $|\nabla h(x,t)| \le C_0' \frac{\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(x)|}}$ and
$|\nabla^2 h(x,t)| \le C_0\theta(x)^{\frac{1}{\beta+1}-1}$. Therefore, $\sigma_0(A) \le h(x,t) \le \sigma_1(A)$, $|\nabla h(x,t)| \le \sigma_1$ and $|\nabla^2 h(x,t)| \le \sigma_1$. From (45), we have $h(x,t_0) = H^*(x)$ for $|x| \ge \frac{\epsilon_0}{6}$. Hence, Proposition 2.5 applies and we get: $|h(x,t) - h(x,t_0)| + |\nabla h(x,t) - \nabla h(x,t_0)| \le \frac{\eta_0}{2} < \eta_0$, which contradicts the hypotheses of Case 3.
This concludes the proof of *i*) and *ii*) of Proposition 2.6.

Proof of iii):
Let us recall that K_0, δ_0 and C_0 are fixed independently of A, where A is taken larger than some $A_7 > 0$, ϵ_0, α_0 and C_0' are fixed in terms of A, and $t_0 \in [t_7(A), T)$, $\eta_0 \le \eta_7(A)$, for some $t_7(A) < T$. and $\eta_7(A) > 0$. Let us prove this lemma:

Lemma 2.7 *There exists $A_8 > 0$ such that for all $A \ge A_8$, there exist $t_8(A) < T$ and $\eta_8(A)$ such that for all $t_0 \in [t_8, T)$ and $\eta_0 \le \eta_8(A)$, and under the hypotheses of Proposition 2.6, the conclusion iii) holds.*

Proof: From lemma 2.5, we have: $\forall t \in [t_0, t_*]$,
$h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ and $(q_0(s_*), q_1(s_*)) \in \partial\hat{V}_A(s_*)$, which means that $q_m(s_*) = \epsilon A s_*^{-2}$ for some $m \in \{0,1\}$ and $\epsilon \in \{-1,1\}$. From (44), the conclusion follows if we show that $\epsilon\frac{dq_m}{ds}(s_*) > 0$.

From (24) and (34), we have: $\int \chi(s_*)\frac{\partial q}{\partial s}(s_*)k_m d\mu = \int \chi(s_*)\mathcal{L}q(s_*)k_m d\mu +$
$\int \chi(s_*)\left[V(s_*)q(s_*) + B(q) + T(q) + R(s_*) + e^{-\frac{ps_*}{p-1}}f_1(e^{\frac{s_*}{p-1}}(\varphi+q))\right]k_m d\mu.$
If we take $t_0 \in [t_{11}(K_0, \epsilon_0(A), A, 0, C_0'), T)$ and $\eta_0 \le \eta_{11}(\epsilon_0(A))$, then we get from lemma 3.2 (see section 3):

$$|\frac{dq_m}{ds}(s_*) - (1-\frac{m}{2})q_m(s_*)| \le \frac{C_6}{s_*^2}$$

for some C_6 independent from all the other constants. Since $q_m(s_*) = \epsilon A s_*^{-2}$, we have $\epsilon\frac{dq_m}{ds}(s_*) > 0$ for $A \ge 4C_6$. ■

Conclusion of the proof: If we take $A = \max(A_7, A_8)$ and $\eta_0 = \min(\eta_7(A), \eta_8(A), \frac{1}{2}\min_{|x|\ge\frac{\epsilon_0}{4}} h(x,t_0))$ ($\min_{|x|\ge\frac{\epsilon_0}{4}} h(x,t_0) > 0$ according to (45) and (5)), and $t_0 = \max(t_7(A), t_8(A))$, then both *i*) and *ii*) of Proposition 2.6 hold. This concludes the proof of Proposition 2.6. Let us note that with this choice, the reduction of the proof of Proposition 2.1 holds. ■

Part II: Topological argument

From Proposition 2.6, we claim that there exist $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$ such that $h(d_0, d_1) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. The proof is similar to the analogous one in [22], let us give its main ideas.

We proceed by contradiction: From $i)$ of Proposition 2.6, we have
$\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$,
$h(d_0, d_1, t_0) \in H \cap S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t_0)$. Therefore, we define for each $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$ a time $t_*(d_0, d_1)$ as being the infinitum of all $t \in [t_0, T)$ such that
$h(d_0, d_1, t) \notin S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$. By $ii)$ of Proposition 2.6, we have
$(q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \in \partial \hat{V}_A(s_*(d_0, d_1))$ where $s_* = -\log(T - t_*)$.

Hence, we can define from (44) the following function:

$$\begin{array}{llll} \Psi : & \mathcal{D}(t_0, K_0, A) & \to & \partial \mathcal{C} \\ & (d_0, d_1) & \to & \frac{s_*(d_0, d_1)^2}{A} (q_0, q_1)(d_0, d_1, s_*(d_0, d_1)) \end{array}$$

where $\mathcal{C}$ is the unit square of $\mathbb{R}^2$.
Now we claim

Proposition 2.7 *i)* Ψ *is a continuous mapping from* $\mathcal{D}(t_0, K_0, A)$ *to* $\partial \mathcal{C}$.
ii) There exists a non trivial affine function $g : \mathcal{D}(t_0, K_0, A) \to \mathcal{C}$ *such that* $\Psi \circ g^{-1}_{|\partial \mathcal{C}} = Id_{|\partial \mathcal{C}}$.

Proof: The proof is very similar to the proof of Proposition 3.3 in [22], that is the reason why we give only the important arguments.

$i)$ follows from the continuity in H of the solution $h(t)$ at a fixed time t with respect to initial data, and the transversality property $iii)$ of Proposition 2.6.

From $ii)$ of lemma 2.4, we have $\forall (d_0, d_1) \in \partial \mathcal{D}(t_0, K_0, A)$, $s_*(d_0, d_1) = s_0$ and $ii)$ follows. ∎

From Proposition 2.7, a contradiction follows (Index Theory). Therefore, there exist $(d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$ such that
$h(d_0, d_1) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0)$. By Proposition 2.1 and the *Conclusion of the proof* of Proposition 2.6, the main Theorem follows.

3 A priori estimates of $u(t)$ in the blow-up zone

This section is devoted to the proof of Proposition 2.3. Let us consider $t_0 < T$, K_0, ϵ_0, α_0, A, δ_0, C_0', C_0 and η_0. We assume that (d_0, d_1) is chosen so that $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$ where $s_0 = -\log(T - t_0)$, and that $\forall t \in [t_0, t_*]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for some $t_* \in [t_0, T)$.

Then we improve some of the bounds given in $i)$ of the Definition of $S^*(t)$ for $h(t_*)$. More precisely, we improve the bounds of $q_2(s_*)$, $q_-(y, s_*)$, $q_e(y, s_*)$ and $\left(\frac{\partial q}{\partial y}\right)_\perp (y, s_*)$ with $s_* = -\log(T - t_*)$.

For this purpose, we consider the equation (24) satisfied by $q(s)$ and the one satisfied by $\frac{\partial q}{\partial y}(s)$ as well as their integral formulations:

$$0 = -\frac{\partial q}{\partial s} + (\mathcal{L} + V(y,s))q + B(q) + T(q) + R(y,s) + \tag{54}$$

$e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q))$
with $\mathcal{L}=\Delta-\frac{1}{2}y.\nabla+1$, $V(y,s)=p\varphi(y,s)^{p-1}-\frac{p}{p-1}$,
$B(q)=(\varphi+q)^p-\varphi^p-p\varphi^{p-1}q$,
$T(q)=-a\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}+a\frac{|\nabla\varphi|^2}{\varphi}$, $R(y,s)=-\frac{\partial\varphi}{\partial s}+\Delta\varphi-\frac{1}{2}y.\nabla\varphi-\frac{\varphi}{p-1}+\varphi^p-a\frac{|\nabla\varphi|^2}{\varphi}$,
$f_1(u)=\alpha^{\frac{\beta}{\beta+1}}u^{1+\frac{1}{\alpha}}F(\alpha^{\frac{1}{\beta+1}}u^{-\frac{1}{\alpha}})-u^p$,
if $r(y,s)=\frac{\partial q}{\partial y}(y,s)$ then

$$\frac{\partial r}{\partial s}=(\mathcal{L}-\frac{1}{2}+V)r+\frac{\partial}{\partial y}(B(q)+T(q))(y,s)+R_1(y,s)$$
$$+e^{-s}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{s}{p-1}}(\varphi+q))$$

with $R_1(y,s)=\frac{\partial R}{\partial y}(y,s)+\frac{\partial V}{\partial y}q(y,s)$,

if $K(s,\sigma)$ and $K_1(s,\sigma)$ are respectively the fundamental solution of $\mathcal{L}+V$ and $\mathcal{L}-\frac{1}{2}+V$ (note that $K_1(s,\sigma)=e^{-\frac{s-\sigma}{2}}K(s,\sigma)$), then for $s\geq\sigma\geq s_0$, $q(s)=$

$$K(s,\sigma)q(\sigma)+\int_\sigma^s d\tau K(s,\tau)(B(q(\tau))+T(q(\tau)))+\int_\sigma^s d\tau K(s,\tau)R(\tau)$$
$$+\int_\sigma^s d\tau K(s,\tau)e^{-\frac{p\tau}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi(\tau)+q(\tau))),\tag{55}$$

and

$$r(s)=K_1(s,\sigma)r(\sigma)+\int_\sigma^s d\tau K_1(s,\tau)(\frac{\partial}{\partial y}(B(q)+T(q))(\tau)+R_1(\tau))$$
$$+\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}(\tau)+r(\tau))f_1'(e^{\frac{\tau}{p-1}}(\varphi(\tau)+q(\tau))).\tag{56}$$

We proceed in two steps: in Step 1, using the fact that $h(t)\in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ for $t\in[T-e^{-\sigma},T-e^{-(\sigma+\rho)}]$ for some $\sigma\geq s_0$ and $\rho\geq 0$, we derive bounds on terms in the right hand side of equation (54) truncated by χ and projected on h_2, and on terms in the right hand sides of equations (55) and (56), expanded respectively as in (35) and (36).

In Step 2, we use these bounds and these equations to find new bounds on q_-, q_e and $r_\perp$ on one hand, and a bound on $\frac{dq_2}{ds}(s)$ on the other hand. This latter bound yields a better estimate on $q_2(s)$ (this estimate is obtained differently from the analogous term in [22] and [25]) .

Step 1: A priori estimates of $q(s)$

We first show that if (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0))\in\hat{V}_A(s_0)$, then $q(s_0)$ is strictly included in $V_{K_0,A}(s_0)$. In other words, at initial time s_0, the finite dimensional variable $(q_0(s_0),q_1(s_0))$ determines the size of the hole function $q(s_0)$. In fact we have an estimate more precise than the one in lemma 2.4:

Lemma 3.1 *For each $A>0$, there exists $s_2(A)>0$ such that for each $s_0\geq s_2(A)$ and $K_0>20$, if $h(x,t_0)$ is given by (45) and (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0))\in\hat{V}_A(s_0)$, then*

$$\begin{array}{llll}|q_2(s_0)| & \leq s_0^{-2}\log s_0, & |q_-(y,s_0)| & \leq Cs_0^{-2}(1+|y|^3),\\ |q_e(y,s_0)| & \leq s_0^{-1/2}, & |r_\perp(y,s_0)| & \leq s_0^{-2}(1+|y|^3),\end{array}$$

and $|r(y,s_0)| \le s_0^{-1/2}$ *for* $|y| \ge K_0\sqrt{s_0}$.

Proof: The proof is included in the proof of lemma 2.4: See the end of its Step 2. ■

Now we consider $\sigma \ge s_0$ and $\rho \in [0,\rho^*]$. We suppose that $\forall t \in [T-e^{-\sigma}, T-e^{-(\sigma+\rho)}]$ $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$. Then we give bounds on terms in right hand sides of equations (54), (55) and (56), expanded as in (34).

Remark: In fact, we give in lemma 3.2 estimates on equation (54) projected on h_m with $m = 0, 1$ or 2. Only $m = 2$ is useful for the proof of Proposition 2.3. The estimates for $m = 0$ or 1 are in a large part the same, they are useful for the proof of Proposition 2.6.

Lemma 3.2 *There exists* $K_{11} > 0$ *and* $A_{11} > 0$ *such that for each* $K_0 \ge K_{11}$, $\epsilon_0 > 0$, $A \ge A_{11}$, $\rho^* > 0$, $C_0' > 0$, *there exists* $t_{11}(K_0, \epsilon_0, A, \rho^*, C_0')$ *with the following property:*

$\forall t_0 \in [t_{11}(K_0,\epsilon_0,A,\rho^*,C_0'), T)$, $\forall \rho \in [0,\rho^*]$, *for all* $\delta_0 \le \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$ *and* $\eta_0 \le \eta_{11}(\epsilon_0)$ *for some* $\eta_{11}(\epsilon_0) > 0$, *assume that*

- $h(x,t_0)$ *is given by (45) and* (d_0,d_1) *is chosen so that* $(q_0(s_0), q_1(s_0)) \in \hat{V}_A(s_0)$

- for some $\sigma \ge -\log(T-t_0)$, *we have* $\forall t \in [T-e^{-\sigma}, T-e^{-(\sigma+\rho)}]$ $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$. *Then,* $\forall s \in [\sigma, \sigma+\rho]$,

I) Equation (54): If $m = 0, 1$ *or* 2,

$$|\int \chi(y,s)k_m(y)\frac{\partial q}{\partial s}(y,s)d\mu - q_m'(s)| \le e^{-s} \tag{57}$$

$$|\int \chi(y,s)k_m(y)\mathcal{L}q(y,s)d\mu - (1-\frac{m}{2})q_m(s)| \le e^{-s} \tag{58}$$

$$|\int \chi(y,s)k_m(y)V(y,s)q(y,s)d\mu| \le s^{-5/2} \tag{59}$$

$$|\int \chi(y,s)k_m(y)B(q)(y,s)d\mu| \le Cs^{-3} \tag{60}$$

$$|\int \chi(y,s)k_m(y)T(q)(y,s)d\mu| \le s^{-2-1/4} \tag{61}$$

$$|\int \chi(y,s)k_m(y)R(y,s)d\mu| \le Cs^{-2} \tag{62}$$

$$|\int \chi(y,s)k_m(y)e^{-\frac{ps}{p-1}}f_1(e^{\frac{s}{p-1}}(\varphi+q))d\mu| \le e^{-s}. \tag{63}$$

If $m = 2$, *then we have more precisely:*

$$|\int \chi(y,s)k_2(y)V(y,s)q(y,s)d\mu + \frac{2p}{s(p-a)}q_2(s)| \le CAs^{-3} \tag{64}$$

$$|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu - \frac{2a}{s(p-a)}q_2(s)| \le CAs^{-3} \tag{65}$$

$$|\int \chi(y,s)k_2(y)R(y,s)d\mu| \le Cs^{-3} \tag{66}$$

II) Equation (55):

Case $\sigma \geq s_0$:

$$(67)\qquad |\alpha_-(y,s)| \leq C(Ae^{-(s-\sigma)/2} + A^2e^{-(s-\sigma)^2})s^{-2}(1+|y|^3)$$
$$(68)\qquad |\alpha_e(y,s)| \leq C(A^2e^{-(s-\sigma)/p} + AK_0^3e^{s-\sigma})s^{-1/2}$$

where $\alpha(s) = K(s,\sigma)q(\sigma)$ is expanded as in (35),

$$(69)\qquad |\beta_-(y,s)| \leq C(s-\sigma)s^{-2}(1+|y|^3)$$
$$(70)\qquad |\beta_e(y,s)| \leq (s-\sigma)s^{-1/2}$$

where $\beta(s) = \int_\sigma^s d\tau K(s,\tau)\,(B(q(\tau)) + T(q(\tau)))$,

$$(71)\qquad |\gamma_-(y,s)| \leq C(s-\sigma)s^{-2}(1+|y|^3)$$
$$(72)\qquad |\gamma_e(y,s)| \leq CK_0^3(s-\sigma)e^{s-\sigma}s^{-1/2}$$
$$(73)$$

where $\gamma(s) = \int_\sigma^s d\tau K(s,\tau)R(\tau)$ is expanded as in (35),

$$(74)\qquad |\delta_-(y,s)| \leq C(s-\sigma)s^{-2}(1+|y|^3)$$
$$(75)\qquad |\delta_e(y,s)| \leq C(s-\sigma)s^{-1/2}$$

where $\delta(s) = \int_\sigma^s d\tau K(s,\tau)e^{-\frac{ps}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi+q))$ is expanded as in (35).
Case $\sigma = s_0$: More precisely,

$$(76)\qquad |\alpha_-(y,s)| \leq Cs^{-2}(1+|y|^3)$$
$$(77)\qquad |\alpha_e(y,s)| \leq CK_0^3e^{s-\sigma}s^{-1/2}.$$

III) Equation (56):
Case $\sigma \geq s_0$:

$$(78)\quad |P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))| \leq C(Ae^{-\frac{(s-\sigma)}{2}} + C(K_0)C_0'e^{-(s-\sigma)^2})\frac{1+|y|^3}{s^2}$$
$$(79)\quad |P_\perp(\chi(s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial}{\partial y}(B(q)+T(q))(\tau))| \leq C(s-\sigma)^{1/2}\frac{1+|y|^3}{s^2}$$
$$(80)\qquad |P_\perp(\chi(s)\int_\sigma^s d\tau K_1(s,\tau)R_1(\tau))| \leq C(s-\sigma)\frac{1+|y|^3}{s^2}$$
$$(81)\qquad |P_\perp(\chi\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y}+r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))|$$

$\leq C(s-\sigma)\frac{1+|y|^3}{s^2}$ *where $P_\perp$ is defined in (34).*
Case $\sigma = s_0$: More precisely,

$$(82)\qquad |P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))| \leq Cs^{-2}(1+|y|^3).$$

Proof: See Appendix B. ■

Step 2: Lemma 3.2 implies Proposition 2.3

Let $K_0 \geq K_{02} > 0$, $\epsilon_0 > 0$, $A \geq A_2(K_0) > 0$ where $A_2(K_0)$ will be fixed later, and $C_0' \leq A^3$. Let $t_0 > 0$ to be fixed in $[t_2(K_0,\epsilon_0,A,C_0'),T)$ (where $t_2(K_0,\epsilon_0,A,C_0')$ will be defined later). Consider $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$ and

$\eta_0 \le \eta_2(\epsilon_0)$. Let $h(d_0,d_1)$ be a solution of equation (II) with initial data (45) defined on $[t_0,t_*]$ with $t_* \in [t_0,T)$, such that
- (d_0,d_1) is chosen so that $(q_0(s_0),q_1(s_0)) \in \hat{V}_A(s_0)$ ($s_0 = -\log(T-t_0)$ and q is defined by (23)),
- $\forall t \in [t_0,t_*]$, $h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,\eta_0,t)$ and $q(s_*) \in \partial V_{K_0,A}(s_*)$.

We want to show that

$$
(83)\quad \begin{array}{rclrcl} |q_2(s_*)| & \le & A^2 s_*^{-2}\log s_* - s_*^{-3}, & |q_-(y,s_*)| & \le & \frac{A}{2}s_*^{-2}(1+|y|^3) \\ |q_e(y,s_*)| & \le & \frac{A^2}{2}s_*^{-1/2}, & |r_\perp(y,s_*)| & \le & \frac{A}{2}s_*^{-2}(1+|y|^3) \end{array}
$$

where

$$
r(y,s) = \frac{\partial q}{\partial y}(y,s).
$$

We consider $\rho_1(K_0,A) \ge \rho_2(K_0,A)$ two positive numbers (which will be fixed later in terms of K_0 and A). The conclusion follows if we treat Case 1 where $s_* - s_0 \le \rho_1$ and then Case 2 where $s_* - s_0 \ge \rho_2$. The proof relies strongly on estimates of lemma 3.2. Therefore, we suppose $K_0 \ge K_{11}$, $A \ge A_{11}$, $C_0' \le A^3$, $t_0 \ge \max(t_{11}(K_0,\epsilon_0,A,\rho_1,C_0'), t_{11}(K_0,\epsilon_0,A,\rho_2,C_0'))$, $s_0 = -\log(T-t_0) \ge \max(\rho_1,\rho_2)$, $\epsilon_0 > 0$, $\delta_0 \le \frac{1}{2}\hat{k}(1)$, $C_0 > 0$ and $\eta_0 \le \eta_{11}(\epsilon_0)$.

Case 1: $s_* - s_0 \le \rho_1(K_0,A)$

We apply lemma 3.2 with A, $\rho^* = \rho_1$, $\rho = s_* - s_0$ and $\sigma = s_0$.

From equation (54) with $m=2$, we obtain: $\forall s \in [s_0,s_*]$, $|q_2'(s) + 2s^{-1}q_2(s)| \le CAs^{-3} + 2e^{-s} \le CAs^{-3}$. Therefore, $\forall s \in [s_0,s_*]$, $|\frac{d}{ds}(s^2 q_2(s))| \le CAs^{-1}$, and then, using $s_* \le 2s_0$ (indeed, $s_* = s_0 + \rho \le s_0 + \rho_1 \le 2s_0$), we obtain $|q_2(s_*)| \le s_*^{-2}s_0^2|q_2(s_0)| + 2A(s_*-s_0)s_*^{-3}$. Using $|q_2(s_0)| \le s_0^{-2}\log s_0$ which follows from lemma 3.1, we get $|q_2(s_*)| \le s_*^{-2}\log s_* + CA(s_*-s_0)s_*^{-3}$. Together with estimates concerning equations (55) and (56) in lemma 3.2, we obtain:

$$
\begin{array}{rcl} |q_2(s_*)| & \le & s_*^{-2}\log s_* + 2C_1 A s_*^{-2} \\ |q_-(y,s_*)| & \le & C_1(1+s_*-s_0)s_*^{-2}(1+|y|^3) \\ |q_e(y,s_*)| & \le & C_1 K_0^3 e^{s_*-s_0}(1+s_*-s_0)s_*^{-1/2} \\ |r_\perp(y,s_*)| & \le & C_1(1+(s_*-s_0)^{1/2}+(s_*-s_0))s_*^{-2}(1+|y|^3) \\ & \le & 2C_1(1+s_*-s_0)s_*^{-2}(1+|y|^3). \end{array}
$$

To have (83), it is enough to have

$$
(84)\quad 1 \le \frac{A^2}{2},\ 2C_1(1+s_*-s_0) \le \frac{A}{2},\ \text{and } C_1K_0^3 e^{s_*-s_0}(1+s_*-s_0) \le \frac{A^2}{2}
$$

on one hand and

$$
(85)\qquad 2C_1 A s_*^{-2} \le \frac{A^2}{2}\frac{\log s_*}{s_*^2} - s_*^{-3}
$$

on the other hand.

If we restrict ρ_1 to satisfy $2C_1(1+\rho_1) \le A/2$ and $C_1K_0^3e^{\rho_1}(1+\rho_1) \le A^2/2$ (which is possible with $\rho_1 = 3/2\log A$ for $A \ge A_6(K_0)$ large enough), then (84) is satisfied, since $s_* - s_0 \le \rho_1$. Now if $s_0 \ge s_6(A)$, then (85) is satisfied. Thus (83) is satisfied also. This concludes Case 1.

Case 2: $s_* - s_0 \geq \rho_2(K_0, A)$

We apply lemma 3.2 with A, $\rho = \rho^* = \rho_2$ and $\sigma = s_* - \rho_2$. From equation (54) with m=2, we obtain $\forall s \in [\sigma, s_*]$, $|q_2'(s) + 2s^{-1}q_2(s)| \leq CAs^{-3}$. Using the same argument as Case 1 and $|q_2(\sigma)| \leq A^2\sigma^{-2}\log\sigma$, and then estimates on equation (55) and (56), we obtain:

$$\begin{array}{rcl} |q_2(s_*)| & \leq & A^2 s_*^{-2}\log(s_* - \rho_2) + 2C_2 A\rho_2 s_*^{-3} \\ |q_-(y, s_*)| & \leq & C_2(Ae^{-\rho_2/2} + A^2 e^{-\rho_2^2} + \rho_2)s_*^{-2}(1 + |y|^3) \\ |q_e(y, s_*)| & \leq & C_2(A^2 e^{-\rho_2/p} + AK_0^3 e^{\rho_2} + K_0^3\rho_2 e^{\rho_2})s_*^{-1/2} \\ |r_\perp(y, s_*)| & \leq & C_2(Ae^{-\rho_2/2} + C(K_0)C_0' e^{-\rho_2^2} + \rho_2{}^{1/2} + \rho_2)s_*^{-2}(1 + |y|^3). \end{array}$$

Since $C_0' \leq A^3$, in order to obtain (83), it is enough to have

$$(86) \quad \begin{array}{rcl} f_{A,\rho_2}(s_*) & \geq & 0 \\ C_2(Ae^{-\rho_2/2} + A^2 e^{-\rho_2^2} + \rho_2) & \leq & \frac{A}{2} \\ C_2(A^2 e^{-\rho_2/p} + AK_0^3 e^{\rho_2} + K_0^3\rho_2 e^{\rho_2}) & \leq & \frac{A^2}{2} \\ C_2(Ae^{-\rho_2/2} + C(K_0)A^3 e^{-\rho_2^2} + \rho_2{}^{1/2} + \rho_2) & \leq & \frac{A}{2} \end{array}$$

with $f_{A,\rho_2}(s_*) = A^2 s_*^{-2}\log s_* - s_*^{-3} - \left[A^2 s_*^{-2}\log(s_* - \rho_2) + 2C_2 A\rho_2 s_*^{-3}\right]$.

We now fix ρ_2 so that $C_2 K_0^3 A e^{\rho_2} = A^2/8$, i.e. $\rho_2 = \log\left(A/(8C_2K_0^3)\right)$. Then, the conclusion follows if A is large enough. Indeed, for all $A > 1$, we write

$$|f_{A,\log\frac{A}{8C_2K_0^3}}(s_*) - s_*^{-3}\left(A^2\log\frac{A}{8C_2K_0^3} - 2C_2 A\log\frac{A}{8C_2K_0^3} - 1\right)| \leq \frac{A^2(\log\frac{A}{8C_2K_0^3})^2}{s^2(s - \log\frac{A}{8C_2K_0^3})^2}.$$

Then we take $A \geq A_7(K_0, C_0')$ for some $A_7(K_0)$ such that

$$\begin{array}{rcl} A^2\log\frac{A}{8C_2K_0^3} - 2C_2A\log\frac{A}{8C_2K_0^3} - 1 & \geq & 1 \\ C_2(A(\frac{A}{8C_2K_0^3})^{-1/2} + A^2 e^{-(\log\frac{A}{8C_2K_0^3})^2} + \log\frac{A}{8C_2K_0^3}) & \leq & \frac{A}{2} \\ C_2(A^2(\frac{A}{8C_2K_0^3})^{-1/p} + AK_0^3\frac{A}{8C_2K_0^3} + K_0^3\log\frac{A}{8C_2K_0^3}\frac{A}{8C_2K_0^3}) & \leq & \frac{A^2}{2} \\ C_2(A(\frac{A}{8C_2K_0^3})^{-1/2} + C(K_0)A^3 e^{-(\log\frac{A}{8C_2K_0^3})^2} + (\log\frac{A}{8C_2K_0^3})^{1/2} & & \end{array}$$

$+\log\frac{A}{8C_2K_0^3}) \leq \frac{A}{2}$.

Afterwards, we take $s_0 \geq s_7(K_0, A)$ so that $\forall s \geq s_0$, $A^2(\log\frac{A}{8C_2K_0^3})^2 s^{-2}(s - \log\frac{A}{8C_2K_0^3})^{-2} \leq s^{-3}/2$.

This way, (86) is satisfied for $A \geq A_7(K_0)$ and $s_0 \geq s_7(K_0, A)$. This concludes Case 2.

We remark that for $A \geq A_8(K_0)$, we have $\rho_1 = \frac{3}{2}\log A \geq \log\frac{A}{8C_2K_0^3} = \rho_2$.

If we take now $K_{02} = K_{11}$, $A_2(K_0) = \max(A_{11}, A_6(K_0), A_7(K_0), A_8(K_0))$ and $t_2 = \max(t_{11}(K_0, \epsilon_0, A, \rho_1(A), C_0'), T - e^{-\rho_1(A)}$, $t_{11}(K_0, \epsilon_0, A, \rho_2(K_0, A), C_0'), T - e^{-\rho_2(K_0,A)}, T - e^{-s_6(A)}, T - e^{-s_7(K_0,A)})$, $\eta_2(\epsilon_0) = \eta_{11}(\epsilon_0)$, then we conclude the proof of Proposition 2.3.

4 A priori estimates in P_2 and P_3

In this section, we estimate directly the solutions of equation (II).

4.1 Estimates in P_2

Let us recall that $\hat{k}(\tau) = \left((\beta+1)(1-\tau) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}$ and that it is defined for $\tau \in [0, \hat{T}]$ with $\hat{T} > 1$.

Proposition 4.1 *There exists $K_{03} > 0$ such that for all $K_0 \geq K_{03}$, $\delta_1 \leq 1$, $\xi_0 \geq 1$ and $C_0^* > 0$, $C_0'^* > 0$, $C_0''^* > 0$ we have the following property: Assume that k is a solution of equation*

$$\frac{\partial k}{\partial \tau} = \Delta k - \frac{1}{k^\beta} \tag{87}$$

for $\tau \in [\tau_1, \tau_2)$ with $0 \leq \tau_1 \leq \tau_2 \leq 1$ $(< \hat{T})$. Assume in addition: $\forall \tau \in [\tau_1, \tau_2]$,
i) $\forall \xi \in [-2\xi_0, 2\xi_0]$, $|k(\xi, \tau_1) - \hat{k}(\tau_1)| \leq \delta_1$ and $|\nabla k(\xi, \tau_1)| \leq \frac{C_0''^*}{\xi_0}$,
ii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $|\nabla k(\xi, \tau)| \leq \frac{C_0'^*}{\xi_0}$ and $|\nabla^2 k(\xi, \tau)| \leq C_0^*$,
iii) $\forall \xi \in [-\frac{7\xi_0}{4}, \frac{7\xi_0}{4}]$, $k(\xi, \tau) \geq \frac{1}{2}\hat{k}(\tau)$. Then, for $\xi_0 \geq \xi_{03}(C_0'^*, C_0^*, C_0''^*)$ there exists $\epsilon = \epsilon(K_0, C_0'^*, \delta_1, \xi_0)$ such that $\forall \xi \in [-\xi_0, \xi_0]$, $\forall \tau \in [\tau_1, \tau_2]$, $|k(\xi, \tau) - \hat{k}(\tau)| \leq \epsilon$ and $|\nabla k(\xi, \tau)| \leq \frac{2C_0''^*}{\xi_0}$, where $\epsilon \to 0$ as $(\delta_1, \xi_0) \to (0, +\infty)$.

Proof: We can assume $\tau_1 = 0$ and $\tau_2 = \tau_0 \leq 1$.
Step 1: Gradient estimate

Lemma 4.1 *Under the assumptions of Proposition 4.1, we have $\forall \xi \in [-\frac{5\xi_0}{4}, \frac{5\xi_0}{4}]$, $\forall \tau \in [0, \tau_0]$ $|\nabla k(\xi, \tau)| \leq \frac{2C_0''^*}{\xi_0}$, if $\xi_0 \geq \xi_{03}(C_0'^*, C_0^*, C_0''^*)$.*

Proof: We have $\forall \xi \in [-2\xi_0, 2\xi_0]$, $\forall \tau \in [0, \tau_0]$,

$$\frac{\partial}{\partial \tau}\nabla k = \Delta(\nabla k) + \beta\frac{\nabla k}{k^{\beta+1}}.$$

From iii), we have for $|\xi| \leq \frac{7\xi_0}{4}$, $|\frac{1}{k^{\beta+1}}| \leq 1$ for K_0 large. If $\theta = |\nabla h|^2$, then, by a direct calculation, $2\frac{\partial k}{\partial \xi}\Delta\left(\frac{\partial k}{\partial \xi}\right) \leq \Delta\theta$ and $\theta_\tau \leq \Delta\theta + C\theta$ for $|x| \leq \frac{7\xi_0}{4}$. Let us consider $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\chi_1(x) = 1$ for $|x| \leq \frac{3\xi_0}{2}$, $\chi_1(x) = 0$ for $|x| \geq \frac{7\xi_0}{4}$, $0 \leq \chi_1 \leq 1$, $|\nabla\chi_1| \leq \frac{C}{\xi_0}$ and $|\Delta\chi_1| \leq \frac{C}{\xi_0^2}$. Then, $\theta_1 = \chi_1\theta$ satisfies the following inequality:
$\theta_{1\tau} \leq \Delta\theta_1 - 2\nabla\chi_1.\nabla\theta - \Delta\chi_1\theta + C\theta_1$
$\leq \Delta\theta_1 + C(C_0'^*, C_0^*)\xi_0^{-2}1_{\{\frac{3\xi_0}{2} \leq |x| \leq 2\xi_0\}} + C\theta_1$. With $\theta_2 = e^{-C\tau}\theta_1$, we have

$$\theta_{2\tau} \leq \Delta\theta_2 + C(C_0'^*, C_0^*)\xi_0^{-2}1_{\{\frac{3\xi_0}{2} \leq |x| \leq 2\xi_0\}} \text{ and } 0 \leq \theta_2(0) \leq \frac{{C_0''^*}^2}{\xi_0^2}.$$

Therefore, by the maximum principle, $\forall \xi \in [-\frac{5\xi_0}{4}, \frac{5\xi_0}{4}]$,$\forall \tau \in [0, \tau_0]$, $\theta(\xi, \tau) \leq \frac{{C_0''^*}^2}{\xi_0^2} + C(C_0'^*, C_0^*)^2\xi_0^{-2}e^{-C'\xi_0^2}$. Hence, for $|\xi| \leq \frac{5\xi_0}{4}$, $\forall \tau \in [0, 1]$, $|\nabla k(\xi, \tau)| \leq \frac{C_0''^*}{\xi_0} + \frac{C(C_0'^*, C_0^*)}{\xi_0}e^{-C'\xi_0^2} \leq \frac{2C_0''^*}{\xi_0}$, if $\xi_0 \geq \xi_{03}(C_0'^*, C_0^*, C_0''^*)$, which yields the conclusion. ■

Step 2: Estimates on k

We are now able to conclude the proof of Proposition 4.1.

Lemma 4.2 *For $|\xi| \leq \xi_0$, $\forall \tau \in [0, \tau_0]$, we have $|k(\xi, \tau) - \hat{k}(\tau)| \leq \epsilon$, where $\epsilon \to 0$ as $\xi_0 \to +\infty$ and $\delta_1 \to 0$.*

Proof: Let us consider k_1 a solution of equation (87) such that $\forall \xi \in [-2, 2]$, $\forall \tau \in [0, \tau_0]$, $|k_1(\xi, 0) - \hat{k}(0)| \leq \delta_1$, $|\nabla k_1(\xi, \tau)| \leq \epsilon$. Let us show that for $|\xi| \leq 2$, $\forall \tau \in [0, \tau_0]$, $|k_1(0, \tau) - \hat{k}(\tau)| \leq C(K_0)\epsilon + \delta_1$ where $C(K_0)$ is independent from ϵ.

We have $\forall \tau \in [0, \tau_0]$, $k_1(0, \tau) = \frac{1}{|B_2(0)|} \int_{|\xi| \leq 2} k_1(\xi, \tau) dx + k_2(\tau)$, and $\frac{1}{k_1(0,\tau)^\beta} = \frac{1}{|B_2(0)|} \int_{|\xi| \leq 2} \frac{1}{k_1(\xi,\tau)^\beta} d\xi + k_3(\tau)$, where $|B_2(0)|$ is the volume of the sphere of radius 2 in $\mathbb{R}^N$, $\|k_2\|_{L^\infty} \leq 2\epsilon$ and $\|k_3\|_{L^\infty} \leq C\epsilon$.

In the distribution sense, for ϵ small enough, considering $\tilde{k}(\tau) = \frac{1}{|B_2(0)|} \int_{|\xi| \leq 2} k_1(\xi, \tau) d\xi$, we have

$$-\frac{1}{\tilde{k}^\beta} - C\epsilon \leq \frac{d\tilde{k}}{d\tau} \leq -\frac{1}{\tilde{k}^\beta} + C\epsilon$$

and $|\tilde{k}(0) - \hat{k}(0)| \leq C\epsilon + \delta_1$.

Together with (87), we obtain by classical a priori estimates that $\forall \tau \in [0, \tau_0]$, $|\tilde{k}(\tau) - \hat{k}(\tau)| \leq C(K_0)\epsilon + \delta_1$ (since $C_1 \leq |\hat{k}(\tau)| \leq C_1'(K_0)$) and therefore $\forall |\xi| \leq 2$, $\forall \tau \in [0, \tau_0]$, $|h_1(0, \tau) - \hat{h}(\tau)| \leq C(K_0)\epsilon + \delta_1$. Applying this result to $h_1(\xi, \tau) = h(\tau, \xi - x_0)$ for $x_0 \in [-\xi_0 + 2, \xi_0 - 2]$, from the assumption and step 1 we obtain lemma 4.2.

Lemmas 4.1 and 4.2 yield Proposition 4.1.

4.2 Estimates in P_3

We claim the following

Proposition 4.2 *For all $\epsilon > 0$, $\epsilon_0 > 0$, $\sigma_0 > 0$, and $\sigma_1 > 0$, there exists $t_4(\epsilon, \epsilon_0, \sigma_0, \sigma_1) < T$ such that $\forall t \in [t_4, T)$, if h is a solution of (II) on $[t_0, t_*]$ for some $t_* \in [t_0, T)$ satisfying*

i) for $|x| \in [\frac{\epsilon_0}{6}, \frac{\epsilon_0}{4}]$, $\forall t \in [t_0, t_*]$,

$$\sigma_0 \leq h(x, t) \leq \sigma_1, \ |\nabla h(x, t)| \leq \sigma_1 \ \textit{and} \ |\nabla^2 h(x, t)| \leq \sigma_1, \tag{88}$$

ii) $h(x, t_0) = H^*(x)$ for $|x| \geq \frac{\epsilon_0}{6}$ where H^* is defined by (5), then for $|x| \in [\frac{\epsilon_0}{4}, +\infty)$, $\forall t \in [t_0, t_*]$,

$$|h(x, t) - h(x, t_0)| + |\nabla h(x, t) - \nabla h(x, t_0)| \leq \epsilon.$$

Proof:

Let us obtain the estimates on h for $|x| \geq \frac{\epsilon_0}{4}$. The estimates on ∇h can be obtained similarly. We argue by contradiction. Let us consider $t_\epsilon \in (t_0, t_*)$ such that

$$\forall t \in [t_0, t_\epsilon), \ \|h(x, t) - h(x, t_0)\|_{L^\infty(|x| \geq \frac{\epsilon_0}{4})} \leq \epsilon \tag{89}$$

and $\|h(x, t_\epsilon) - h(x, t_0)\|_{L^\infty(|x| \geq \frac{\epsilon_0}{4})} = \epsilon$.

We can assume $\epsilon \leq \frac{1}{4} \min_{|x| \geq \frac{\epsilon_0}{6}} H^*(x)$. We can remark that (5) implies that

$|h(x,t_0)| = H^*(x) > C_0(\epsilon_0) > 0$ for $|x| \geq \frac{\epsilon_0}{6}$, therefore, we have
$|F(h(x,t))| \leq C(\epsilon_0)$ for $|x| \geq \frac{\epsilon_0}{6}$ and $t \in [t_0, t_\epsilon)$.
From assumption $i)$, we have in fact $\forall t \in [t_0, t_\epsilon]$, for $\frac{\epsilon_0}{6} \leq |x| \leq \frac{\epsilon_0}{4}$, $h(x,t) \geq \sigma_0 > 0$ and $|F(h(x,t))| \leq C(\sigma_0)$. We then consider $h_1(x,t) = \chi_1(x)h(x,t)$ where $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^N, [0,1])$, $\chi_1 \equiv 1$ for $|x| \geq \frac{\epsilon_0}{5}$, $\chi_1 \equiv 0$ for $|x| \leq \frac{\epsilon_0}{6}$, $|\nabla \chi_1| \leq \frac{C}{\epsilon_0}$ and $|\Delta \chi_1| \leq \frac{C}{\epsilon_0^2}$. We then have:

$$\frac{\partial h_1}{\partial t} = \Delta h_1 - 2\nabla \chi_1.\nabla h - \Delta \chi_1 h - \chi_1 F(h).$$

Since $\forall t \in [t_0, t_*]$, $|2\nabla\chi_1.\nabla h| + |\Delta \chi_1 h| \leq C(\epsilon_0, \sigma_1)1_{\{\frac{\epsilon_0}{6} \leq |x| \leq \frac{\epsilon_0}{5}\}}(x)$, we write

$$\frac{\partial h_1}{\partial t} = \Delta h_1 + \tilde{f}_1(x,t) - \chi_1 F(h)$$

with $|\tilde{f}_1(x,t)| \leq C(\epsilon_0, \sigma_1)1_{\{\frac{\epsilon_0}{6} \leq |x| \leq \frac{\epsilon_0}{5}\}}(x)$.

Let us now consider the case of a bounded domain Ω and the case $\Omega = \mathbb{R}^N$, since there is a small difference in the proof.

i) Ω is a bounded domain:

In this case,
$\forall t \in [t_0, t_\epsilon)$, $h_1(t) - S(t-t_0)h_1(t_0) = \int_{t_0}^t ds S(t-s)[\tilde{f}_1(x,t) - \chi_1 F(h)]$ where $S(.)$ is the linear heat flow. Hence,
$|h_1(t) - h_1(t_0)|_{L^\infty} \leq |h_1(t) - S(t-t_0)h_1(t_0)|_{L^\infty} + |S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty} \leq$
$\int_{t_0}^t ds[|S(t-s)\tilde{f}_1(s)|_{L^\infty} + |S(t-s)C(\epsilon_0,\sigma_0)\chi_1 F(h)|_{L^\infty}]$
$+|S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty}$
$\leq \int_{t_0}^t ds[\frac{ds}{\sqrt{t-s}}|\tilde{f}_1(s)|_{L^N} + |S(t-s)C(\epsilon_0,\sigma_0)1_{\{\Omega\}}|_{L^\infty}]$
$+|S(t-t_0)h_1(t_0) - h_1(t_0)|_{L^\infty}$
$\leq C(\epsilon_0,\sigma_0,\sigma_1)\sqrt{t-t_0} + |S(t-t_0)\chi_1 H^* - \chi_1 H^*|_{L^\infty}$.
Now, if $t_0 \in [t_5(\epsilon,\epsilon_0,\sigma_0,\sigma_1), T)$, then we have $|h_1(t_\epsilon) - h_1(t_0)|_{L^\infty} \leq \frac{\epsilon}{2}$, which is a contradiction with (89).

Therefore, $\forall t \in [t_0, t_*]$ $|h(x,t) - h(x,t_0)|_{L^\infty(|x| \geq \frac{\epsilon_0}{4})} \leq \epsilon$.

ii) Case $\Omega = \mathbb{R}^N$: we define $h_2(x,t) = \psi(x) + h_1(x,t)$ where $\psi(x)$ is introduced in the introduction (such that $\psi \in C^\infty(\mathbb{R}^N)$, $\psi \equiv 0$ on $[-1,1]$, $\psi(x) = a_1|x|$ for $|x| \geq 2$). From the fact that $\frac{\partial h_2}{\partial t} = \Delta h_2 + F(h_2(x) + \psi(x)) + \Delta\psi$ and that for $|v| \geq 1$, $|F(v)| + |F'(v)| \leq Ce^{-v}$, we obtain using similar techniques:
$\forall t \in [t_0, t_*)$, $|h_2(x,t) - h_2(x,t_0)|_{L^\infty} \leq \epsilon$ or equivalently: $\forall t \in [t_0, t_*)$,
$|h_1(x,t) - h_1(x,t_0)|_{L^\infty} \leq \epsilon$. This concludes the proof of Proposition 4.2.

A Proof of lemma 2.4

We must show that for suitable $(d_0, d_1) \in \mathbb{R}^2$, the estimates of the Definition of $S^*(t)$ hold for $t = t_0$. Since estimate $iii)$ holds obviously, we show in a first step that $h(t_0) \in H$ and estimate $ii)$ holds, for all choices of (d_0, d_1), provided that t_0 is near T. Then, in step 2, we find $\mathcal{D}(t_0, K_0, A)$ such that $\forall (d_0, d_1) \in \mathcal{D}(t_0, K_0, A)$, $q(s_0) \in V_{K_0,A}(s_0)$, where q is the function introduced in (23).

Step 1: Estimate $ii)$ of the Definition of $S^*(t)$

Let us first remark that from (45), (5) and (6), we have $h(t_0) \in \psi + H^1 \cap W^{2,\infty}(\mathbb{R})$. Moreover, one can see from (45), (10), (27) and (5) that $\forall x \in \mathbb{R}$, $h(x,t_0) \geq C(t_0,d_0,d_1,\epsilon_0) > 0$. Therefore, $h(t_0) \in H$.

Let us consider $t_0 < T$, K_0, ϵ_0, α_0, δ_1, and C_1, and show that if these constants are suitably chosen, then for $|x| \in [\frac{K_0}{4}\sqrt{(T-t_0)|\log(T-t_0)}, \epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$$|k(x,\xi,\frac{t_0-t(x)}{\theta(x)}) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)| \leq \delta_1,\ |\frac{\partial k}{\partial \xi}| \leq \frac{C^*(K_0)}{\sqrt{|\log\theta(x)|}}, \tag{90}$$

and $|\nabla_\xi^2 k| \leq C_1$ where k, $\hat{k}$, $t(x)$ and $\theta(x)$ are defined in (28), (30) and (29).

Let us first introduce some useful notations:

$$\theta_0 = T - t_0,\ r(t_0) = \frac{K_0}{4}\sqrt{\theta_0|\log\theta_0|} \text{ and } R(t_0) = \theta_0^{\frac{1}{2}}|\log\theta_0|^{\frac{p}{2}}, \tag{91}$$

and remark that thanks to (31), we have for fixed K_0:

$$\theta(r(t_0)) \sim \theta_0,\ \theta(R(t_0)) \sim \frac{16}{K_0^2}\theta_0|\log\theta_0|^{p-1}, \quad \theta(2R(t_0)) \sim \frac{64}{K_0^2}\theta_0|\log\theta_0|^{p-1},$$

$$\log\theta(r(t_0)) \sim \log\theta(R(t_0)) \sim \log\theta(2R(t_0)) \sim \log\theta_0 \text{ as } t_0 \to T. \tag{92}$$

If $\alpha_0 \leq \frac{K_0}{16}$ and $\epsilon_0 \leq \frac{2}{3}C(a_1,\beta)$, then it follows from (29) that for $|x| \in [r(t_0),\epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, we have $|\xi\sqrt{\theta(x)}| \leq \frac{|x|}{2}$ and

$$\frac{r(t_0)}{2} \leq \frac{|x|}{2} = |x| - \frac{|x|}{2} \leq |x+\xi\sqrt{\theta(x)}| \leq \frac{3}{2}|x| \leq \frac{3}{2}\epsilon_0 \leq C(a_1,\beta). \tag{93}$$

Therefore, we get from (28), (45), and (27): for $|x| \in [r(t_0),\epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$, $k(x,\xi,\frac{t_0-t(x)}{\theta(x)}) =$

$$(I)\chi_1(x+\xi\sqrt{\theta(x)},t_0) + (II)(1-\chi_1(x+\xi\sqrt{\theta(x)},t_0)) \tag{94}$$

with $(I) = \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}}\hat{\Phi}(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}})$ and $(II) = \theta(x)^{-\frac{1}{\beta+1}}H^*(x+\xi\sqrt{\theta(x)})$.

Estimate on k:

By linearity and (46), it is enough to prove that for $|x| \in [r(t_0),2R(t_0)]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$\left|(I) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right| \leq \frac{\delta_1}{2} \tag{95}$$

and for $|x| \in [R(t_0),\epsilon_0]$ and $|\xi| \leq 2\alpha_0\sqrt{|\log\theta(x)|}$,

$$\left|(II) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right| \leq \frac{\delta_1}{2}. \tag{96}$$

We begin with (95). From (4) and (30), we have:

$$\left|(I) - \hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right| = |\left((\beta+1)(\frac{\theta_0}{\theta(x)}) + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta(x)|\log\theta_0|}\right)^{\frac{1}{\beta+1}}$$
$$-\left((\beta+1)(\frac{\theta_0}{\theta(x)}) + \frac{(\beta+1)^2}{4\beta}\frac{K_0^2}{16}\right)^{\frac{1}{\beta+1}}| \leq C|\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta(x)|\log\theta_0|} - \frac{K_0^2}{16}|^{\frac{1}{\beta+1}}$$

$\le CK_0^{\frac{2}{\beta+1}}\left||\sqrt{\frac{\log\theta(x)}{\log\theta_0}}+\frac{4\xi}{K_0\sqrt{|\log\theta_0|}}|^2-1\right|^{\frac{1}{\beta+1}}.$

Since $|x|\in[r(t_0),R(t_0)]$ and $|\xi|\le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$\left(\sqrt{\frac{\log(\theta(R(t_0)))}{\log\theta_0}}(1-4\frac{\alpha_0}{K_0})\right)^2-1\le\left(\sqrt{\frac{\log\theta(x)}{\log\theta_0}}+\frac{4\xi}{K_0\sqrt{\log\theta_0}}\right)^2-1$

$$\le\left(\sqrt{\frac{\log(\theta(r(t_0)))}{\log\theta_0}}(1+4\frac{\alpha_0}{K_0})\right)^2-1. \tag{97}$$

From (97) and (92), we find $\alpha_5(K_0,\delta_1)$ and $t_5(K_0,\delta_1)<T$ such that $\forall\alpha_0\le\alpha_5$, $\forall t_0\in[t_5,T)$,

$|(I)-\hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)|\le CK_0^{\frac{2}{\beta+1}}||\sqrt{\frac{\log\theta(x)}{\log(T-t_0)}}+\frac{4\xi}{K_0\sqrt{|\log(T-t_0)|}}|^2-1|^{\frac{1}{\beta+1}}\le\frac{\delta_1}{2}.$

Now, we treat (96). Let $|x|\in[R(t_0),\epsilon_0]$ and $|\xi|\le 2\alpha_0\sqrt{|\log\theta(x)|}$. We have from (94), (5), (29) and (30),

$(II)=\left[\frac{(\beta+1)^2|x+\xi\sqrt{\theta(x)}|^2}{8\beta\theta(x)|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}=\left[\frac{(\beta+1)^2|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi|^2}{8\beta|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}$ and

$\left|(II)-\hat{k}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right|$

$=\left|\left[\frac{(\beta+1)^2|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi|^2}{8\beta|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}-\left[(\beta+1)\left(\frac{\theta_0}{\theta(x)}\right)+\frac{(\beta+1)^2K_0^2}{64\beta}\right]^{\frac{1}{\beta+1}}\right|$

$\le\left|\frac{(\beta+1)^2}{8\beta}\left(\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi\right|^2}{|\log|x+\xi\sqrt{\theta(x)}||}-\frac{K_0^2}{8}\right)-(\beta+1)\left(\frac{\theta_0}{\theta(x)}\right)\right|^{\frac{1}{\beta+1}}$

$\le C\left[(I_1)+(I_2)\right]$

with $(I_1)=\left|\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\xi\right|^2}{|\log|x+\xi\sqrt{\theta(x)}||}-\frac{K_0^2}{8}\right|^{\frac{1}{\beta+1}}$ and $(I_2)=\left|\frac{\theta_0}{\theta(x)}\right|^{\frac{1}{\beta+1}}$.

Let us bound (I_1). Since $|\xi|\le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have from (29),

$|(I_1)|\le\left|\frac{\left|\frac{K_0}{4}\sqrt{|\log\theta(x)|}+\alpha_0\sqrt{|\log\theta(x)|}\right|^2}{|\log|x+\alpha_0\sqrt{\theta(x)|\log\theta(x)|}||}-\frac{K_0^2}{8}\right|^{\frac{1}{\beta+1}}$

$=\left|\frac{\log\theta(x)}{\log x+\log(1+\frac{4\alpha_0}{K_0})}\left(\frac{K_0}{4}+\alpha_0\right)^2-\frac{K_0^2}{8}\right|^{\frac{1}{\beta+1}}.$

Since $|x|\le\epsilon_0$ and $\log\theta(x)\sim 2\log|x|$ as $x\to 0$ (see (31)), we find $\alpha_6(K_0,\delta_1)$ such that for each $\alpha_0\le\alpha_6(K_0,\delta_1)$, there is $\epsilon_6(K_0,\delta_1,\alpha_0)$ such that for all $\epsilon_0\le\epsilon_6(K_0,\delta_1,\alpha_0)$, for $|x|\in[R(t_0),\epsilon_0]$ and $|\xi|\le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have

$$|(I_1)|\le\frac{\delta_1}{2}. \tag{98}$$

Let us bound (I_2). Since $|x|\ge R(t_0)$, we have from (92), $|(I_2)|\le\left|\frac{\theta_0}{\theta(R(t_0))}\right|^{\frac{1}{\beta+1}}$ $\le C(K_0)|\log\theta_0|^{-\frac{(p-1)}{\beta+1}}$. Therefore, if $t_0\ge t_6(K_0,\delta_1)$, then

$$|(I_2)|\le\frac{\delta_1}{2}. \tag{99}$$

Combining (98) and (99), we get: If $\alpha_0 \le \alpha_6(K_0,\delta_1)$, $\epsilon_0 \le \epsilon_6(K_0,\delta_1,\alpha_0)$ and $t_0 \ge t_6(K_0,\delta_1)$, then for $|x| \in [R(t_0),\epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, (96) holds.

Estimate on $\frac{\partial k}{\partial \xi}$:

From (94), we have for $|x| \in [r(t_0),\epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, $\frac{\partial k}{\partial \xi}(x,\xi,\frac{t_0-t(x)}{\theta(x)}) = E_1+E_2+E_3$ where $E_1 =$

$$(100)\quad \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}} \frac{\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}}\nabla\hat{\Phi}\left(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}}\right)\chi_1(x+\xi\sqrt{\theta(x)},t_0),$$

$$(101)\qquad E_2 = \theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}}\nabla H^*(x+\xi\sqrt{\theta(x)})(1-\chi_1(x+\xi\sqrt{\theta(x)},t_0)),$$

$$(102)\qquad E_3 = E_4\theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}}\frac{\partial\chi_1}{\partial x}(x+\xi\sqrt{\theta(x)},t_0) \text{ with}$$

$$E_4 = \theta_0^{\frac{1}{\beta+1}}\hat{\Phi}\left(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}}\right) - H^*(x+\xi\sqrt{\theta(x)}).$$

In order to get the estimate on $\frac{\partial k}{\partial \xi}$, it is enough to show that for

$$(103)\ |x| \in [r(t_0),2R(t_0)] \text{ and } |\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|},\ |E_1| \le \frac{C(K_0)}{\sqrt{|\log\theta(x)|}},$$

$$(104)\quad |x| \in [R(t_0),\epsilon_0] \text{ and } |\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|},\ |E_2| \le \frac{C(K_0)}{\sqrt{|\log\theta(x)|}},$$

$$(105)\ |x| \in [R(t_0),2R(t_0)] \text{ and } |\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|},\ |E_3| \le \frac{C(K_0)}{\sqrt{|\log\theta(x)|}}.$$

We begin with E_1. Let $|x| \in [r(t_0),2R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$. From (4), it follows that $|\nabla\hat{\Phi}(z)| \le C|z|^{\frac{1-\beta}{\beta+1}}$. Therefore, by (100),
$|E_1| \le \left(\frac{\theta_0}{\theta(x)}\right)^{\frac{1}{\beta+1}} \frac{\sqrt{\theta(x)}}{\sqrt{\theta_0|\log\theta_0|}} \frac{|x+\xi\sqrt{\theta(x)}|^{\frac{1-\beta}{\beta+1}}}{(\theta_0|\log\theta_0|)^{\frac{1-\beta}{2(\beta+1)}}}$
$\le |\log\theta_0|^{-\frac{1}{\beta+1}}\theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}}C(\beta)|x|^{\frac{1-\beta}{\beta+1}}$ (by (93))
$\le C(K_0)|\log\theta(x)|^{-\frac{1}{2}}|\log\theta_0|^{-\frac{1}{\beta+1}}|\log\theta(x)|^{\frac{1}{\beta+1}}$ (by (29))
$\le C(K_0)|\log\theta(x)|^{-\frac{1}{2}}|\log\theta_0|^{-\frac{1}{\beta+1}}|\log\theta\,(r(t_0))\,|^{\frac{1}{\beta+1}}$ (since $|x| \ge r(t_0)$)
$\le C(K_0)|\log\theta(x)|^{-\frac{1}{2}}$ for $t_0 \ge t_7(K_0)$ (use (92)), which implies (103).

Now we treat E_2. Let $|x| \in [R(t_0),\epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$. From (101), we have $|E_2| \le \theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}}|\nabla H^*(x+\xi\sqrt{\theta(x)})|$
$\le \theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}}|\nabla H^*(\gamma x)|$ with $\gamma = \frac{3}{2}$ is $\beta \le 1$ and $\gamma = \frac{1}{2}$ if $\beta > 1$ (use (93) and (5)). According to lemma 2.2,
$|\nabla H^*(\gamma x)| \sim C(K_0)\frac{\theta(\gamma x)^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(\gamma x)|}} \sim C'(K_0)\frac{\theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}}}{\sqrt{|\log\theta(x)|}}$ as $x \to 0$. This implies (104) for $\epsilon \le \epsilon_7(K_0)$.

Now we show the bound on E_3. We consider $|x| \in [R(t_0),2R(t_0)]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, and find a bound on E_4. From (102),

$E_4 = \left[(\beta+1)\theta_0 + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0|}\right]^{\frac{1}{\beta+1}} - \left[\frac{(\beta+1)^2}{8\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log|x+\xi\sqrt{\theta(x)}||}\right]^{\frac{1}{\beta+1}}$. From (93) and (91), we have

$$\alpha(t_0) \le (\beta+1)\theta_0 + \frac{(\beta+1)^2}{4\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0|} \le C\alpha(t_0)$$

$$\text{and } \alpha(t_0) \le \frac{(\beta+1)^2}{8\beta}\frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log|x+\xi\sqrt{\theta(x)}||} \le C\alpha(t_0)$$

with $\alpha(t_0) \sim C\theta_0|\log\theta_0|^{p-1}$. Therefore, $|E_4| \le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}$ $\times\left|(\beta+1)\theta_0 + \frac{(\beta+1)^2}{8\beta}|x+\xi\sqrt{\theta(x)}|^2\left(\frac{2}{|\log\theta_0|} - \frac{1}{|\log|x+\xi\sqrt{\theta(x)}||}\right)\right|$ $\le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}\left|\theta_0 + \frac{|x+\xi\sqrt{\theta(x)}|^2}{|\log\theta_0||\log|x+\xi\sqrt{\theta(x)}||}\left|\log\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta_0}\right|\right|$ $\le C\left[\theta_0|\log\theta_0|^{p-1}\right]^{-\frac{\beta}{\beta+1}}\left|\theta_0 + \theta_0|\log\theta_0|^{p-2}\log\log\theta_0\right|$ (use (93), (91) and $|x| \in [R(t_0), 2R(t_0)]$). Hence

$$|E_4| \le C\theta_0^{\frac{1}{\beta+1}}|\log\theta_0|^{-\frac{(p-1)\beta}{\beta+1}}\left[1 + |\log\theta_0|^{p-2}\log\log\theta_0\right]. \tag{106}$$

Using (46) and (27), we have

$$\left|\frac{\partial\chi_1}{\partial x}\right| \le C\theta_0^{-\frac{1}{2}}|\log\theta_0|^{-\frac{p}{2}}. \tag{107}$$

From (92) and the fact that $|x| \in [R(t_0), 2R(t_0)]$, we have: $\theta(x)^{\frac{1}{2}-\frac{1}{\beta+1}} \le \theta(\delta R(t_0))^{\frac{1}{2}-\frac{1}{\beta+1}} \le C(K_0)\left[\theta_0|\log\theta_0|^{p-1}\right]^{\frac{1}{2}-\frac{1}{\beta+1}}$ if $t_0 \ge t_8(K_0)$, with $\delta = 2$ if $\beta \ge 1$ and $\delta = 1$ if $\beta < 1$.

Combining this with (102), (106) and (107), we get $|E_3| \le C(K_0)|\log\theta_0|^{-p+\frac{1}{2}}\left[1 + |\log\theta_0|^{p-2}\log\log\theta_0\right] \le |\log\theta_0|^{-\frac{1}{2}}$ if $t_0 \ge t_8(K_0)$.

Since $\log\theta_0 \sim \log\theta(R(t_0))$ as $t_0 \to T$ (see (92)) and $R(t_0) \le |x|$, this yields (105) for $t_0 \ge t_9(K_0)$.

The expected bound (90) on $\frac{\partial k}{\partial\xi}$ follows from (103), (104) and (105).

Estimate on Δk:

In the same way, we show that if $t_0 \ge t_{10}(K_0, \epsilon_0, C_1)$, then for $|x| \in [r(t_0), \epsilon_0]$ and $|\xi| \le 2\alpha_0\sqrt{|\log\theta(x)|}$, we have $|\nabla_\xi^2 k(x, \xi, \frac{t-t(x)}{\theta(x)})| \le C_1$.

Step 2: Estimate $i)$ **of the Definition of** $S^*(t)$

From (23) and (45), we have $\chi(y, s_0)q(y, s_0) =$

$$(d_0 + d_1\frac{y}{\sqrt{s_0}})\chi_0(\frac{|y|}{\sqrt{s_0}K_0/16}) - \frac{\kappa}{2(p-a)s_0}\chi_0(\frac{|y|}{\sqrt{s_0}K_0}). \tag{108}$$

Using (34), (26), (25) and simple calculations, and taking $K_0 \ge 20$, we have: if $t_0 \ge t_{11}$, then

$$\begin{aligned} q_0(s_0) &= d_0 \int\chi_0(\tfrac{|y|}{\sqrt{s_0}K_0/16})d\mu - \tfrac{\kappa}{2(p-a)s_0}\int\chi_0(\tfrac{|y|}{\sqrt{s_0}K_0})d\mu, \\ q_1(s_0) &= \tfrac{d_1}{\sqrt{s_0}} \int\tfrac{y^2}{2}\chi_0(\tfrac{|y|}{\sqrt{s_0}K_0/16})d\mu, \end{aligned} \tag{109}$$

and

$$(110)\qquad q_0(s_0) = d_0(1+O(e^{-s_0})) - \frac{\kappa}{2(p-a)s_0} + O(e^{-s_0})$$

$$(111)\qquad q_1(s_0) = \frac{d_1}{\sqrt{s_0}}(1+O(e^{-s_0}))$$

$$(112)\qquad q_2(s_0) = d_0 O(e^{-s_0}) + O(e^{-s_0}),$$

$|q_-(y,s_0)| \le Ce^{-s_0}(1+|d_0|)(1+|y|^2) + C|d_1|s_0^{-\frac{1}{2}}e^{-s_0}|y|$
$+\frac{\kappa}{2(p-a)s_0}(1-\chi_0(\frac{|y|}{K_0\sqrt{s_0}})) + (|d_0|+|d_1\frac{y}{\sqrt{s_0}}|)(1-\chi_0(\frac{|y|}{\sqrt{s_0}K_0/16}))$.
Since $\forall n \in \mathbb{N}$, $|\chi_0(z)-1| \le C_n|z|^n$, and $K_0 \ge 20$, we get

$$(113)\qquad |q_-(y,s_0)| \le (|d_0|+|d_1|+\frac{C}{s_0})\frac{(1+|y|^3)}{s_0^{3/2}}.$$

Let us show that

$$(114)\qquad |q_e(y,s_0)| \le \frac{C}{s_0}.$$

From (23), we have $q_e(y,s_0) = Q_1 + Q_2$ where $Q_2 = \frac{\kappa}{2(p-a)s_0}(1-\chi(y,s)) \le Cs_0^{-1}$ and $Q_1 = (1-\chi(y,s))\left[\frac{e^{-\frac{s_0}{p-1}}\alpha^{\frac{\alpha}{\beta+1}}}{h(x,t_0)^\alpha} - \Phi(\frac{y}{\sqrt{s_0}})\right]$ with $x = ye^{-s_0/2}$ and $t_0 = T - e^{-s_0}$.
If $|x| \le R(t_0)$ (see (91) for $R(t_0)$), then we have from (45), (46) and (27) $Q_1 = 0$.
If $|x| \ge R(t_0)$, then we have from (10), (45), (91), (9) and easy calculations: $\Phi(\frac{y}{\sqrt{s_0}}) \le \Phi(\frac{e^{\frac{s_0}{2}}R(t_0)}{\sqrt{s_0}}) \le Cs_0^{-1}$ and
$h(x,t_0) \ge \chi_1(x,t_0)(T-t_0)^{\frac{1}{\beta+1}}C\left[\Phi(\frac{e^{\frac{s_0}{2}}R(t_0)}{\sqrt{s_0}})\right]^{-\frac{1}{\alpha}} + (1-\chi_1(x,t_0))H^*(R(t_0))$
$\ge C(T-t_0)^{\frac{1}{\beta+1}}s_0^{\frac{1}{\alpha}}$.
Therefore, by (9), $|Q_1| \le Cs_0^{-1}$, which yields (114).

By analogous calculations, one can easily obtain:

$$(115)\qquad |\left(\frac{\partial q}{\partial y}\right)_\perp(y,s_0)| \le C\frac{(|d_0|+|d_1|+1/s_0)}{\sqrt{s_0}}\frac{(1+|y|^3)}{s_0^{3/2}}$$

and $|\frac{\partial q}{\partial y}(y,s_0)| \le s_0^{-1}$ for $|y| \ge K_0\sqrt{s_0}$.

From (109), one sees that $g : (d_0,d_1) \to (q_0(s_0), q_1(s_0))$ is an affine function. Let us introduce $\mathcal{D}(t_0,K_0,A) = g^{-1}\left([-\frac{A}{s_0^2},\frac{A}{s_0^2}]^2\right)$. $\mathcal{D}(t_0,K_0,A)$ is obviously a rectangle.

If $(d_0,d_1) \in \mathcal{D}(t_0,K_0,A)$, or equivalently $|q_m(s_0)| \le \frac{A}{s_0^2}$ for $m = 0,1$, then, from (110) and (111), we obtain $|d_0| \le Cs_0^{-1}$ and $|d_1| \le CAs_0^{-3/2}$. Combining this with (112), (113), (114) and (115), we obtain $\forall A > 0$, there exists $t_{12}(A) < T$ such that for each $t_0 \in [t_{12},T)$:

$$\begin{array}{llll} |q_2(s_0)| & \le s_0^{-2}\log s_0, & |q_-(y,s_0)| & \le Cs_0^{-2}(1+|y|^3), \\ |q_e(y,s_0)| & \le s_0^{-1/2}, & |\left(\frac{\partial q}{\partial y}\right)_\perp(y,s_0)| & \le s_0^{-2}(1+|y|^3), \\ |\frac{\partial q}{\partial y}(y,s_0)| & \le s_0^{-1} & \text{for } |y| \ge K_0\sqrt{s_0} & \end{array}$$

and $q(s_0) \in V_{K_0,A}(s_0)$.

Now, putting the conclusions of Steps 1 and 2 together and taking $K_{01} = 20$, $\alpha_1(K_0,\delta_1) = \min\left(\frac{K_0}{4}, \alpha_5(K_0,\delta_1), \alpha_6(K_0,\delta_1)\right)$, $\epsilon_1(K_0,\delta_1,\alpha_0)$ $= \min\left(\frac{2}{3}C(a_1,\beta), \epsilon_6(K_0,\delta_1,\alpha_0), \epsilon_7(K_0)\right)$, $t_1(K_0,\delta_1,\epsilon_0,A,C_1) =$ $\max\left(t_5(K_0,\delta_1), t_6(K_0,\delta_1), t_7(K_0), t_8(K_0), t_9(K_0), t_{10}(K_0,\epsilon_0,C_1), t_{11}, t_{12}(A)\right)$, we reach the conclusion of lemma 2.4 *i*). *ii*) is obviously true by construction and by (109). ■

B Proof of lemma 3.2

We start with some technical results on equations (54), (55) and (56) (Step 1). In Step 2, we conclude the proof of lemma 3.2.

Step 1: Estimates on equations (54), (55) and (56)

i) Sizes of q and ∇q:

Lemma B.1 *For all $K_0 \geq 1$ and $\epsilon_0 > 0$, there exists $t_1(K_0,\epsilon_0)$ such that $\forall t_0 \in [t_1,T)$, for all $A \geq 1$, $\alpha_0 > 0$, $C_0 > 0$, $C_0' > 0$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$ and $\eta_0 \leq \eta_1(\epsilon_0)$ for some $\eta_1(\epsilon_0) > 0$, we have the following property: Assume that $h(x,t_0)$ is given by (45) and that for some $t \in [t_0,T)$, we have $h(t) \in S^*(t_0,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0',C_0,t)$, then:*
i) $|q(y,s)| \leq CA^2K_0^3 s^{-1/2}$ and $|q(y,s)| \leq CA^2 s^{-2}\log s(1+|y|^3)$,
ii) $|\nabla q(y,s)| \leq C(K_0,C_0')A^2 s^{-1/2}$, $|\nabla q(y,s)| \leq C(K_0,C_0')A^2 s^{-2}\log s(1+|y|^3)$, $|(1-\chi(y,s))\nabla q(y,s)| \leq C(K_0)C_0' s^{-\frac{1}{2}}$, where $s = -\log(T-t)$ and q is defined in (23).

Proof:
i): From *i*) of the definition of $S^*(t)$, we have $q(s) \in V_{K_0,A}(s)$. Therefore, the proof of lemma 3.8 in [22] holds.
ii): Arguing similarly as for *i*), we obtain from *i*) of the definition of $S^*(t)$ and (26):

$$|\chi(y,s)\nabla q(y,s)| \leq CA^2\frac{\log s}{s^2}(1+|y|^3) \text{ and } |\chi(y,s)\nabla q(y,s)| \leq C\frac{A^2K_0^3}{\sqrt{s}}.$$

Since $|\nabla\varphi(y,s)| \leq Cs^{-1/2}$ and $s^{-1/2} \leq s^{-2}|y|^3$ for $|y| \geq K_0\sqrt{s}$ and $K_0 \geq 1$, we have to prove that $|(1-\chi(y,s))\nabla(q+\varphi)(y,s)| \leq C(K_0)C_0' s^{-1/2}$ in order to conclude the proof.

From (23), this reduces to show that $\forall t \geq t_0$, for $|x| \geq r(t)$,

$$|\nabla u|(x,t) = C(\alpha)\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \leq C(K_0,C_0')\frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}} \tag{116}$$

$$\text{where} \qquad r(t) = K_0\sqrt{(T-t)|\log(T-t)|}. \tag{117}$$

Let us consider two cases:

Case 1: $|x| \in [r(t), \epsilon_0]$. We use the information contained in $ii)$ of the definition of $S^*(t)$. From (28), we have

$$h(x,t) = \theta(x)^{\frac{1}{\beta+1}} k(x, 0, \tau(x,t)) \tag{118}$$

$$\text{and} \qquad \nabla_x h(x,t) = \theta(x)^{\frac{1}{\beta+1}-\frac{1}{2}} \nabla_\xi k(x, 0, \tau(x,t)) \tag{119}$$

with $\tau(x,t) = \frac{t-t(x)}{\theta(x)}$. Therefore, since $p = \frac{\alpha+\beta+1}{\alpha}$,

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) = \theta(x)^{-(\frac{1}{p-1}+\frac{1}{2})} \frac{|\nabla_\xi k|}{k^{\alpha+1}}(x, 0, \tau(x,t)). \tag{120}$$

Using the definition of $S^*(t)$, we have for $|x| \in [r(t), \epsilon_0]$

$$|k(x,0,\tau(x,t)) - \hat{k}(\tau)| \le \delta_0 \text{ and } |\nabla_\xi k(x,0,\tau(x,t))| \le \frac{C_0'}{\sqrt{|\log \theta(x)|}}. \tag{121}$$

Since $\delta_0 \le \frac{1}{2}\hat{k}(1)$, (120) and (29) yield for $|x| \in [r(t), \epsilon_0]$:

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \le C(K_0) C_0' \frac{\theta(x)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log \theta(x)|}} \le C(K_0) C_0' \frac{\theta\,(r(t))^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log \theta\,(r(t))\,|}} \tag{122}$$

with $C(K_0) = \frac{C}{\hat{k}(0)^{\alpha+1}}$. Since $r(t) \to 0$ as $t \to T$ (see (117)), we have from (31)

$$\theta(r(t)) \sim \frac{2}{K_0^2} \frac{r(t)^2}{|\log r(t)|} \text{ and } \log \theta(r(t)) \sim \log r(t) \text{ as } t \to T. \tag{123}$$

Using (117), we get

$$\frac{(\theta(r(t)))^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(\theta(r(t)))|}} \sim C_4 \frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}} \text{ as } t \to T$$

for some constant C_4. Therefore, if $t_0 \in [t_2(K_0), T)$ for some $t_2(K_0) < T$, then we have for $t \ge t_0$

$$\frac{(\theta(r(t)))^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(\theta(r(t)))|}} \le 2C_4 \frac{(T-t)^{-(\frac{1}{p-1}+\frac{1}{2})}}{\sqrt{|\log(T-t)|}}. \tag{124}$$

Using (122) and (124), we find (116) for $|x| \in [r(t), \epsilon_0]$, provided that $t_0 \in [t_2(K_0), T)$.

Case 2: $|x| \ge \epsilon_0$. We use here the information contained in $iii)$ of the definition of $S^*(t)$, which asserts that

$$|h(x,t) - h(x,t_0)| \le \eta_0 \text{ and } |\nabla h(x,t) - \nabla h(x,t_0)| \le \eta_0$$

for $|x| \ge \epsilon_0$. Let $\eta_1(\epsilon_0) = \frac{1}{2}\min\{\min_{|x|\ge\epsilon_0} |h(x,t_0)|, \min_{|x|\ge\epsilon_0} |\nabla h(x,t_0)|\}$. According to (45) and (5), we have $\eta_1(\epsilon_0) > 0$. If $\eta_0 \le \eta_1(\epsilon_0)$, we get for $|x| \ge \epsilon_0$:

$$\frac{|\nabla h|}{h^{\alpha+1}}(x,t) \le C \frac{|\nabla h|}{h^{\alpha+1}}(x,t_0) = C \frac{|\nabla H^*|}{H^{*(\alpha+1)}}(x)$$

from (45). Therefore, proving (116) for all $t \geq t_0$ reduces to prove it for $t = t_0$. From (5), one easily remarks that $\frac{|\nabla H^*|}{H^{*(\alpha+1)}}(x) \leq C(\epsilon_0)$ for $|x| \geq \epsilon_0$. Therefore, if $t_0 \in [t_4(\epsilon_0), T)$ for some $t_4(\epsilon_0) < T$, then we get (116) for $t = t_0$.

This concludes the proof of (116) for $t = t_0$ and $|x| \geq \epsilon_0$, hence for $t \geq t_0$ and $|x| \geq \epsilon_0$. Thus, with $t_1(K_0, \epsilon_0) = \max(t_2(K_0), t_4(\epsilon_0))$, this concludes the proof of (116) and the proof of lemma B.1. ■

ii) Estimates on K and K_1:

As we remarked before, $K_1(s, \sigma) = e^{-(s-\sigma)/2} K(s, \sigma)$. Hence, any estimate on K holds for K_1 with the adequate changes.

Since K_1 is the fundamental solution of $\mathcal{L} - 1/2 + V$ and $\mathcal{L} - 1/2$ is conjugated to the harmonic oscillator $e^{-x^2/8}(\mathcal{L} - 1/2)e^{x^2/8} = \partial^2 - x^2/16 + 1/4 + 1/2$, we give a Feynman-Kac representation for K_1:

$$K_1(s, \sigma, y, x) = e^{(s-\sigma)(\mathcal{L}-1/2)}(y, x) E(s, \sigma, y, x) \tag{125}$$

where

$$E(s, \sigma, y, x) = \int d\mu_{yx}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} V(\omega(\tau), \sigma+\tau) d\tau} \tag{126}$$

and $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s - \sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s - \sigma) = y$, i.e. the Gaussian probability measure with covariance kernel $\Gamma(\tau, \tau') = \omega_0(\tau)\omega_0(\tau')$

$$+2(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)-\tau'+\tau|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau'-\tau|}, \tag{127}$$

which yields $\int d\mu_{yx}^{s-\sigma} \omega(\tau) = \omega_0(\tau)$ with
$\omega_0(\tau) = (\sinh \frac{s-\sigma}{2})^{-1}(y \sinh \frac{\tau}{2} + x \sinh \frac{s-\sigma-\tau}{2})$.

We have in addition

$$e^{\theta(\mathcal{L}-1/2)}(y, x) = \frac{e^{\theta/2}}{\sqrt{4\pi(1 - e^{-\theta})}} \exp[-\frac{(ye^{-\theta/2} - x)^2}{4(1 - e^{-\theta})}]. \tag{128}$$

Using this formulation for K_1, we give estimates on the dynamics of K and K_1 in the following lemma:

Lemma B.2 *i) $\forall s \geq \tau \geq 1$ with $s \leq 2\tau$, $\int |K(s, \tau, y, x)|(1+|x|^m)dx \leq e^{s-\tau}(1+|y|^m)$.*

ii) There exists $K_2 > 0$ such that for each $K_0 \geq K_2$, $A' > 0$, $A'' > 0$, $A''' > 0$, $\rho^* > 0$, there exists $s_2(K_0, A', A'', A''', \rho^*)$ with the following property: $\forall s_0 \geq s_2$, assume that for $\sigma \geq s_0$, $q(\sigma)$ is expanded as in (35) and satisfies

$$\begin{array}{llllll} |q_m(\sigma)| & \leq & A'\sigma^{-2}, m = 0, 1, & |q_2(\sigma)| & \leq & A''(\log \sigma)\sigma^{-2}, \\ |q_-(y, \sigma)| & \leq & A'''(1 + |y|^3)\sigma^{-2}, & |q_e(y, \sigma)| & \leq & A''\sigma^{-\frac{1}{2}}, \end{array}$$

then, $\forall s \in [\sigma, \sigma + \rho^*]$

$$\begin{array}{lll} |\alpha_-(y, s)| & \leq & C(e^{-\frac{1}{2}(s-\sigma)}A''' + A''e^{-(s-\sigma)^2})(1 + |y|^3)s^{-2}, \\ |\alpha_e(y, s)| & \leq & C(A''e^{-\frac{(s-\sigma)}{p}} + A'''K_0^3 e^{(s-\sigma)})s^{-\frac{1}{2}}, \end{array}$$

where $\alpha(y, s) = K(s, \sigma)q(\sigma)$ is expanded as in (35).

iii) There exists $K_3 > 0$ such that for each $K_0 \geq K_3$, $A' > 0$, $A'' > 0$, $A''' > 0$, $A'''' > 0$, $\rho^* > 0$, there exists $s_3(K_0, A', A'', A''', A'''', \rho^*)$ with the following property: $\forall s_0 \geq s_3$, assume that for $\sigma \geq s_0$, $r(\sigma)$ is expanded as in (36) and satisfies

$$\begin{array}{llllll} |r_0(\sigma)| & \leq & A'\sigma^{-2}, & |r_1(\sigma)| & \leq & A''(\log\sigma)\sigma^{-2}, \\ |r_-(y,\sigma)| & \leq & A'''(1+|y|^3)\sigma^{-2}, & |r_e(y,\sigma)| & \leq & A''''\sigma^{-\frac{1}{2}}, \end{array}$$

then, $\forall s \in [\sigma, \sigma+\rho^*]$

$$|P_\perp(\chi(s)K_1(s,\sigma)r(\sigma))| \leq C(e^{-\frac{1}{2}(s-\sigma)}A''' + A''''e^{-(s-\sigma)^2})(1+|y|^3)s^{-2}.$$

Proof: See corollary 3.1 in [22] for i). See Lemma 3.5 in [22] for ii).

Since $K_1(s,\sigma) = e^{-(s-\sigma)/2}K(s,\sigma)$, and ii) and iii) have similar structure, one can adapt without difficulty the proof of ii) (given in [22]) to get iii). ■

iii) Estimates on $B(q)$:

Lemma B.3 *$\forall K_0 \geq 1$, $\forall A \geq 1$, $\exists s_5(K_0, A)$ such that $\forall s \geq s_5(A, K_0)$, $q(s) \in V_{K_0,A}(s)$ implies $|\chi(y,s)B(q(y,s))| \leq C(K_0)|q|^2$ and $|B(q)| \leq C|q|^{\bar{p}}$ with $\bar{p} = \min(p, 2)$.*

Proof: See Lemma 3.6 in [22]. ■

iv) Estimates on $T(q)$:

Lemma B.4 *For all $K_0 \geq 1$, $A \geq 1$ and $\epsilon_0 > 0$, there exists $t_6(K_0, \epsilon_0, A) < T$ and $\eta_6(\epsilon_0)$ such that for each $t_0 \in [t_6(K_0, \epsilon_0, A), T)$, $\alpha_0 > 0$, $C_0' > 0$, $\delta_0 \leq \frac{1}{2}\hat{k}(1)$, $C_0 > 0$ and $\eta_0 \leq \eta_6(\epsilon_0)$:*
if $h(x, t_0)$ is given by (45) and $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, \eta_0, t)$ for some $t \in [t_0, T)$, then

$$|\chi(y,s)(T(q) + 2a\frac{\nabla\varphi}{\varphi}.\nabla q)| \leq C(K_0, A)(\frac{|y|^2}{s^2}|q| + \frac{|q|^2}{s} + |\nabla q|^2), \tag{129}$$

$$|\chi(y,s)T(q)| \leq C(K_0,A)\chi(y,s)\left(s^{-1}|q| + s^{-1/2}|\nabla q|\right) \tag{130}$$

$$|(1-\chi(y,s))T(q)| \leq C(K_0, C_0')\min(s^{-1}, s^{-5/2}|y|^3) \tag{131}$$

where $s = -\log(T-t)$ and q is defined in (23).

Proof:

Proof of (129) and (130): They both follow from the Taylor expansion of $F(\theta) = -\frac{|\nabla\varphi+\theta\nabla q|^2}{\varphi+\theta q} + \frac{|\nabla\varphi|^2}{\varphi}$ for $\theta \in [0,1]$. Let us compute
$F'(\theta) = q\frac{|\nabla\varphi+\theta\nabla q|^2}{(\varphi+\theta q)^2} - 2\frac{\nabla q.(\nabla\varphi+\theta\nabla q)}{\varphi+\theta q}$,
$F''(\theta) = -2q^2\frac{|\nabla\varphi+\theta\nabla q|^2}{(\varphi+\theta q)^3} + 4q\frac{\nabla q.(\nabla\varphi+\theta\nabla q)}{(\varphi+\theta q)^2} - 2\frac{|\nabla q|^2}{\varphi+\theta q}$.
From $F(1) = F(0) + F'(0) + \int_0^1(1-\theta)F'(\theta)d\theta$, we write

$$\chi(y,s)T(q) = a\chi(y,s)(q\frac{|\nabla\varphi|^2}{\varphi^2} - 2\nabla q.\frac{\nabla\varphi}{\varphi}) + a\int_0^1(1-\theta)\chi(y,s)F''(\theta)d\theta.$$

Using (23), lemma B.1 and (26), we claim that for $s_0 \geq s_7(A, K_0)$, $\forall s \geq s_0$, $\forall \theta \in [0,1]$, $|\nabla\varphi| \leq Cs^{-\frac{1}{2}}$, $\frac{|\nabla\varphi|^2}{\varphi^2} \leq C\frac{|y|^2}{s^2}$ and
$|\chi(y,s)F''(\theta)| \leq C(K_0, A)\chi(y,s)(s^{-1}|q|^2 + |\nabla q|^2)$
$\leq C(K_0, A)(s^{-1}|q| + s^{-\frac{1}{2}}|\nabla q|)$. Therefore, (129) and (130) follow.

Proof of (131): From (23), we have $\frac{|\nabla\varphi|^2}{\varphi}(y,s) \leq Cs^{-1}$. Therefore, if $K_0 \geq 1$, (26) implies that $(1-\chi(y,s))\frac{|\nabla\varphi|^2}{\varphi}(y,s) \leq \min(Cs^{-1}, Cs^{-5/2}|y|^3)$. In order to prove (131), it then remains to prove that
$(1-\chi(y,s))\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}(y,s) \leq \min(Cs^{-1}, Cs^{-5/2}|y|^3)$, or simply, for $|y| \geq K_0\sqrt{s}$, $\frac{|\nabla\varphi+\nabla q|^2}{\varphi+q}(y,s) \leq Cs^{-1}$, since $Cs^{-1} \leq Cs^{-5/2}|y|^3$ for $|y| \geq K_0\sqrt{s}$, if $K_0 \geq 1$.
From (23), this reduces to show that $\forall t \geq t_0$, for $|x| \geq r(t)$,

$$\frac{|\nabla u|^2}{u}(x,t) = C(\alpha)\frac{|\nabla h|^2}{h^{\alpha+2}}(x,t) \leq C(K_0, C_0')\frac{(T-t)^{-\frac{p}{p-1}}}{|\log(T-t)|} \tag{132}$$

where $r(t)$ is introduced in (117). The proof of (132) is in all its steps completely analogous to the proof of (116) given during the course of the proof of lemma B.1, that is the reason why we escape it here. ■

v) Estimates on $R(y,\tau)$:

Lemma B.5 *$\forall y \in \mathbb{R}$, $\forall s \geq 1$, $|R(y,s)| \leq Cs^{-1}$,*
$|R(y,s) - C_1(p,a)s^{-2}| \leq Cs^{-3}(1+|y|^4)$ for some $C_1(p,a) \in \mathbb{R}$, and
$|\frac{\partial R}{\partial y}(y,s)| \leq Cs^{-1-\bar{p}}(|y|+|y|^3)$ where $\bar{p} = \min(p,2)$.

Proof: From (54), we have

$$R(y,s) = -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y.\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p - a\frac{|\nabla\varphi|^2}{\varphi} \text{ where}$$

$$\varphi(y,s) = \Phi^p + \frac{\alpha}{s},\ \Phi = (p-1+bz^2)^{-\frac{1}{p-1}},\ b = \frac{(p-1)^2}{4(p-a)},\ z = \frac{y}{\sqrt{s}}, \tag{133}$$

$\alpha = \frac{\kappa}{2(p-a)}$ and $\kappa = (p-1)^{-\frac{1}{p-1}}$. Therefore,

$$\begin{array}{rcl} R(y,s) &=& -\frac{bz^2}{(p-1)s}\Phi^p + \frac{\alpha}{s^2} - \frac{2b}{(p-1)s}\Phi^p + \frac{4pb^2z^2}{(p-1)^2s}\Phi^{2p-1} \\ &-& \Phi^p - \frac{\alpha}{(p-1)s} + \varphi^p - \frac{4ab^2z^2}{(p-1)^2s}\frac{\Phi^{2p}}{\varphi}. \end{array} \tag{134}$$

Proof of $|R(y,s)| \leq Cs^{-1}$: It follows form (134), and the fact that $|z|^2\Phi^{p-1} + \Phi \leq C$, $\varphi^{-1} \leq \Phi^{-1}$ and $|\Phi^p - \varphi^p| \leq C\alpha s^{-1}$.

Proof of $|R(y,s) - C_1(p,a)s^{-2}| \leq Cs^{-3}(1+|y|^4)$: If $|z| \geq 1$, then $1 \leq s^{-1}|y|^2$ and $|R(y,s) - C_1(p,a)s^{-2}| \leq Cs^{-1} \times \left(s^{-1}|y|^2\right)^2 \leq Cs^{-3}(1+|y|^4)$.
Let us focus on the case $|z| \leq 1$. The method we use consists in expanding each term of (134) in terms of powers of s^{-1} and z^2. From (133), one can easily obtain the following bounds: for $|z| \leq 1$, $\forall s \geq 1$,
$|\Phi^p - \kappa^p + \frac{pb\kappa}{(p-1)^3}z^2| \leq Cz^4$, $|\Phi^{2p-1} - \kappa^{2p-1}| \leq Cz^2$,
$|\varphi^p - \Phi^p - \frac{p\alpha}{s}\Phi^{p-1} - \frac{p(p-1)\alpha^2}{2s^2}\Phi^{p-2}| \leq Cs^{-2}$, $|\Phi^{p-1} - \frac{1}{p-1} + \frac{b}{(p-1)^2}z^2| \leq Cz^4$,

$|\Phi^{p-2} - \kappa^{p-2}| \le Cz^2$ and $|\frac{\Phi^{2p}}{\varphi} - \Phi^{2p-1}| \le Cs^{-1}$.
Combining all these bounds with (134) and (133), and using $|z| \le 1$, we get the result.

Proof of $|\frac{\partial R}{\partial y}(y,s)| \le Cs^{-1-\bar{p}}(|y| + |y|^3)$: The proof is completely similar to the above estimates. We just give its main steps. First, use (134) to compute $\frac{\partial R}{\partial y}$. Then, show that $\forall y \in \mathbb{R}$, $\forall s \ge 1$, $|\frac{\partial R}{\partial y}(y,s)| \le Cs^{\frac{1}{2}-\bar{p}}$, in the same way as for $|R(y,s)| \le Cs^{-1}$. Therefore, if $|z| \ge 1$, this gives the expected bound. If $|z| \le 1$, expand all the terms with respect to s and z^2 to conclude. ■

vi) Estimates on f_1:

Lemma B.6 $\forall u \ge 0$, $|f_1(u)| + |f_1'(u)| \le C$.

Proof: According to (24), (H2) and (19), we have:
$f_1(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - u^p$, $f_1'(u) = -F'(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) - pu^{p-1}$,
$\forall v \in (0,1]$, $F(v) = v^{-\beta}$ $\forall v \ge 1$, $|F(v)| \le Ce^{-v} \le C$. Therefore,
- if $\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}} \le 1$, then $f_1(u) = f_1'(u) = 0$,
- if $\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}} \ge 1$, then $u \le \alpha^{\frac{\alpha}{\beta+1}}$ and $|f_1(u)| + |f_1'(u)| \le C(\alpha)$. ■

Step 2: Conclusion of the proof

Here, we use the lemmas of Step 1 in order to conclude the proof. Therefore, we assume that $K_0 \ge \max(1, K_2, K_3)$, $\epsilon_0 > 0$, $A \ge 1$, $t_0 \ge \max(t_1(K_0,\epsilon_0),$
$T - \exp(-s_2(K_0, A, A^2, A, \rho^*)), T - \exp(-s_2(K_0, C, C, C, \rho^*)),$
$T - \exp(-s_2(K_0, A, 1, C, \rho^*)), T - \exp(-s_3(K_0, CA, CA^2, CA, C(K_0)C_0', \rho^*)),$
$T - \exp(-s_3(K_0, CA, C, 1, 1, \rho^*)), T - \exp(-s_5(K_0, A)), t_6(K_0, \epsilon_0, A))$,
$\delta_0 \le \frac{1}{2}\hat{k}(1)$, $\alpha_0 > 0$, $C_0 > 0$, $C_0' \ge 0$, $\eta_0 \le \min(\eta_1(\epsilon_0), \eta_6(\epsilon_0))$.

We consider $\sigma \ge -\log(T - t_0)$ and $\rho \le \rho^*$, and suppose that $\forall t \in [T - e^{-\sigma}, T - e^{-(\sigma+\rho)}]$, $h(t) \in S^*(t_0, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0', C_0, t)$. Using the definition of $S^*(t)$, and the lemmas of Step 1, we start the proof of the estimates of lemma 3.2.

Below, $O(f)$ stands for a function bounded by f and not by Cf. We use the notations introduced in (34).

I) Equation (54)

Since $q_m'(s) = \frac{d}{ds}\int \chi(y,s)k_m(y)q(y,s)d\mu = \int \frac{\partial}{\partial s}(\chi q)k_m d\mu$, we obtain:
$|\int \chi(y,s)k_m(y)\frac{\partial q}{\partial s}(y,s)d\mu - q_m'(s)| = |\int \frac{\partial \chi}{\partial s}(y,s)k_m(y)q(y,s)d\mu|$
$\le C\frac{A^2K_0^3}{s^{1/2}}e^{-2s}$ by lemma B.1, (25) and (26). If $s_0 \ge s_{12}(K_0, A)$, then (57) follows.

Since $\mathcal{L}$ is self adjoint and $\mathcal{L}k_m = (1-\frac{m}{2})k_m$, there exist two polynomials P_m and Q_m such that $|\int \chi(y,s)k_m(y)\mathcal{L}q(y,s)d\mu - (1-\frac{m}{2})q_m(s)| = |\int[\mathcal{L}(\chi(s)k_m) - \chi(s)k_m]q(s)d\mu| = |\int(\frac{\partial \chi}{\partial y}P_m(y) + \frac{\partial^2\chi}{\partial y^2}Q_m(y))q(s)d\mu|$
$\le CA^2K_0^3s^{-1/2}e^{-2s}$ by lemma B.1, (25) and (26). Therefore,
$|\int \chi(y,s)k_m(y)\mathcal{L}q(y,s)d\mu| \le e^{-s}$ if $s_0 \ge s_{13}(K_0, A)$, which yields (58).

From (54), $|V(y,s)| \le Cs^{-1}(1+|y|^2)$. Therefore,
$|\int \chi(y,s)k_m(y)V(y,s)d\mu| \le CA^2s^{-3}\log s \le s^{-5/2}$ for $s \ge s_{34}(A)$, by lemma B.1 and (25). This yields (59).

From lemmas B.3 and B.1, and (25), we have
$|\int \chi(y,s)k_m(y)B(q)(y,s)d\mu| \le C(K_0)A^4s^{-4}(\log s)^2$.
Now, if $s_0 \ge s_{15}(K_0, A)$, then (60) follows.

By lemmas B.4 and B.1, and (25), we write:
$|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu| \le s^{-2-1/4}$ for $s_0 \ge s_{36}(K_0,A)$, which is (61).
From (54), $|V(y,s) + 2p/(s(p-a))k_2| \le Cs^{-2}(1+|y|^4)$. Since $|q_0(s)| + |q_4(s)| \le CAs^{-2}$ follows from $q(s) \in V_{K_0,A}(s)$, and since $\int \chi(s)k_2^2 q(s)d\mu = q_2(s) + c_0 q_0(s) + c_4 q_4(s)$, we get (64) for $s_0 \ge s_7(A)$.

From lemma B.5, we have $|R(y,s)| \le C(s^{-2} + s^{-3}|y|^4)$. Using (25), we get (62).

From lemma B.6, we have $|e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi+q))| \le Ce^{-\frac{ps}{p-1}}$. Therefore, as before, $|\int \chi(y,s)k_m(y)e^{-\frac{ps}{p-1}} f_1(e^{\frac{s}{p-1}}(\varphi+q))d\mu \le Ce^{-\frac{ps}{p-1}} \le e^{-s}$ for s large and (63) follows.

From (54), $|V(y,s) + 2p/(s(p-a))k_2| \le Cs^{-2}(1+|y|^4)$. Since $|q_0(s)| + |q_4(s)| \le CAs^{-2}$ follows from $q(s) \in V_{K_0,A}(s)$, and since $\int \chi(s)k_2^2 q(s)d\mu = q_2(s) + c_0 q_0(s) + c_4 q_4(s)$, we get (64) for $s_0 \ge s_7(A)$.

By lemmas B.4 and B.1, and (25), we write:
$|\int \chi(y,s)k_2(y)T(q)(y,s)d\mu + E| \le s^{-3}$ for $s_0 \ge s_{16}(K_0,A,C_0')$, where
$E = a/4\int \nabla q(y,s)(\chi(y,s)\frac{\nabla\varphi}{\varphi}(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy$
$-a/4\int q(y,s)\nabla.(\chi(y,s)\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy$
$= O(e^{-s}) - a/4\int q(y,s)\chi(y,s)\nabla.(\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi})dy.$
By simple calculation,
$|\nabla.(\nabla\varphi/\varphi(y^2-2)e^{-|y|^2/4}/\sqrt{4\pi}) - (h_2(y)+h_4(y)/4)/(s(p-a)).e^{-|y|^2/4}/\sqrt{4\pi}|$
$\le P(|y|)e^{-|y|^2/4}/s^2$ where P is a polynomial. Hence $E = O(CA^2 s^{-4}\log s) - a/(4s(p-a))(8q_2(s)+c_4q_4(s)) = O(CAs^{-3}) - 2a/(s(p-a))q_2(s)$ and (65) holds.

(66) follows from lemma B.5, (26) and (25).

II) Equation (55)
(67) and (68) follow from lemma B.2 *ii*) applied with $A' = A''' = A$ and $A'' = A^2$.

Lemmas B.3 and B.1 yield
$|B(q(x,\tau))| \le C|q(x,\tau)|^{\bar p} \le CA^{2\bar p}\tau^{-2\bar p}(\log\tau)^{\bar p}(1+|x|^3)^{\bar p}$.
Lemmas B.4 and B.1 yield
$|T(q(x,\tau))| \le |\chi(x,\tau)T(q(x,\tau))| + |(1-\chi(x,\tau))T(q(x,\tau))|$
$\le C(K_0,A)\tau^{-5/2}\log\tau(1+|x|^3) + C(K_0,C_0')\tau^{-5/2}|x|^3$.
Therefore, $|B(q(\tau)) + T(q(\tau))| \le$

$$C(K_0,A,C_0')\left\{\frac{(\log\tau)^{\bar p}}{\tau^{2\bar p}}(1+|x|^{3\bar p}) + \frac{\log\tau}{\tau^{5/2}}(1+|x|^3)\right\}. \tag{135}$$

This way, $|\beta(y,s)| = |\int_\sigma^s d\tau K(s,\tau)\,(B(q(\tau)) + T(q(\tau)))\,|$
$\le \int_\sigma^s d\tau \int dx|K(s,\tau,y,x)|\,|B(q(x,\tau)) + T(q(\tau))|$
$\le C(K_0,A,C_0')\int_\sigma^s d\tau\,\{\tau^{-2\bar p}(\log\tau)^{\bar p}\int dx|K(s,\tau,y,x)|(1+|x|^{3\bar p})$
$+\tau^{-5/2}\log\tau\int dx|K(s,\tau,y,x)|(1+|x|^3)\}$
$\le C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\,\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^{3\bar p}) + s^{-5/2}\log s(1+|y|^3)\}$
if $s_0 \ge \rho^*$ (Indeed, $s \le \sigma+\rho \le \sigma+\rho^* \le \sigma+s_0 \le 2\sigma \le 2\tau$, and lemma B.2 applies). Hence,
$|\chi(y,s)\beta(y,s)| \le C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\,\{s^{-2\bar p}(\log s)^{\bar p}(1+|y|^3|y|^{3\bar p-3})$

$+s^{-5/2}\log s(1+|y|^3)\}$
$\leq C(K_0,A,C_0')(s-\sigma)e^{s-\sigma}\left\{s^{-2\bar{p}}(\log s)^{\bar{p}}(1+|y|^3(K_0\sqrt{s})^{3\bar{p}-3})\right.$
$+s^{-5/2}\log s(1+|y|^3)\}\leq(s-\sigma)s^{-2}(1+|y|^3)$, if $s_0\geq s_{17}(K_0,A,\rho^*,C_0')$ (use $\bar{p}>1$). This yields $|\beta_m(s)|\leq C(s-\sigma)s^{-2}$ for $m=0,1,2$ and then (69).

Lemmas B.3 and B.1 yield $|B(q(x,\tau))|\leq C|q(x,\tau)|^{\bar{p}}\leq CK_0^{3\bar{p}}A^{2\bar{p}}\tau^{-\bar{p}/2}$.
Lemmas B.4 and B.1 yield
$|T(q(x,\tau))|\leq C(K_0,A)\tau^{-1}+C(K_0,C_0')\tau^{-1}$.
Therefore, $|B(q(\tau))+T(q(\tau))|\leq C(K_0,A,C_0')\tau^{-\bar{p}/2}$.
This way, $|\int_\sigma^s d\tau K(s,\tau)(B(q(\tau))+T(q(\tau)))|$
$\leq\int_\sigma^s d\tau\int dx|K(s,\tau,y,x)||B(q(x,\tau))+T(q(x.\tau))|$
$\leq C(K_0,A,C_0')\int_\sigma^s d\tau\tau^{-\bar{p}/2}\int dx|K(s,\tau,y,x)|$
$\leq C(K_0,A,C_0')s^{-\bar{p}/2}(s-\sigma)e^{s-\sigma}$ if $s_0\geq\rho^*$ (Indeed, $s\leq 2\tau$ and lemma B.2 applies). Hence $|\beta_e(y,s)|\leq C(K_0,A,C_0')s^{-\bar{p}/2}(s-\sigma)e^{\rho^*}\leq(s-\sigma)s^{-1/2}$ if $s_0\geq s_{18}(A,\rho^*,C_0')$ (use $\bar{p}>1$). This yields (70).

Lemma B.5 implies that $\forall\tau>1$, $\forall x\in\mathbb{R}$, $|R_m(\tau)|\leq C\tau^{-2}$, $m=0,1$, $|R_2(\tau)|\leq C\tau^{-2}\log\tau$, $|R_-(x,\tau)|\leq C\tau^{-2}(1+|x|^3)$ and $|R_e(x,\tau)|\leq C\tau^{-1/2}$. Applying lemma B.2 *ii)* with $A'=A''=A'''=C$ and then integrating with respect to $\tau\in[\sigma,s]$ yields (71) and (72).

From lemma B.6, we have $|e^{-\frac{p\tau}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi+q))|\leq Ce^{-\frac{p\tau}{p-1}}$. Therefore, $|\delta(y,s)|=|K(s,\tau)e^{-\frac{p\tau}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi+q))|\leq Ce^{s-\tau}e^{-\frac{p\tau}{p-1}}$ according to *i)* of lemma B.2. Hence,
$|\int_\sigma^s K(s,\tau)e^{-\frac{p\tau}{p-1}}f_1(e^{\frac{\tau}{p-1}}(\varphi+q))|\leq C(s-\sigma)e^{s-\sigma}e^{-\frac{p\sigma}{p-1}}$
$\leq C(s-\sigma)e^{\rho^*}e^{-\frac{p}{p-1}\frac{s}{2}}$ if $s_0\geq\rho^*$,
$\leq(s-\sigma)s^{-2}$ if $s\geq s_{19}(A,\rho^*)$. As before, this implies (74) and (75).

From lemma 3.1 we have $|q_m(s_0)|\leq As_0^{-2}$, $m=0,1$,
$|q_2(s_0)|\leq s_0^{-2}\log s_0$, $|q_-(y,s_0)|\leq Cs_0^{-2}(1+|y|^3)$ and $|q_e(y,s_0)|\leq s_0^{-1/2}$. If we apply lemma B.2 *ii)* with $A'=A$, $A''=1$, $A'''=C$, then (76) and (77) follow.

III) Equation (56)

From definition 34, we have for $m=0,1$,
$r_m(\sigma)=\int\nabla q(y,\sigma)\chi(y,\sigma)k_m(y)d\mu$
$=-\int q(y,\sigma)\nabla(\chi(y,\sigma)k_m e^{-y^2/4}/\sqrt{4\pi})dy$
$=O(e^{-\sigma})-\int q(y,\sigma)\chi(y,\sigma)\nabla(k_m e^{-y^2/4}/\sqrt{4\pi})dy$
$=O(e^{-\sigma})+(m+1)\int q(y,\sigma)\chi(y,s)k_{m+1}(y)d\mu=O(e^{-\sigma})+(m+1)q_{m+1}(\sigma)$.
Hence, if $\sigma\geq s_0\geq s_{21}$, then $|r_0(\sigma)|\leq CA\sigma^{-2}$ and $|r_1(\sigma)|\leq CA^2\sigma^{-2}\log\sigma$. We have $|r_\perp(y,\sigma)|\leq A\sigma^{-2}(1+|y|^3)$ since $q(\sigma)\in V_{K_0,A}(\sigma)$ (see the definition of $S^*(t)$), and $|r_e(y,\sigma)|\leq C(K_0)C_0'\sigma^{-1/2}$ by lemma B.1. Now, we apply lemma B.2 *iii)* with $A'=A'''=CA$, $A''=CA^2$ and $A''''=C(K_0)C_0'$ to conclude the proof of (78)

Estimate (79) is harder than estimate (78) because it involves a parabolic estimate on the kernel K_1.

Setting $I(x,\tau)=B(q(x,\tau))+T(q(x,\tau))$, we write
$K_1(s,\tau)\frac{\partial}{\partial y}(B(q)+T(q))(\tau)=\int dxe^{(s-\tau)(\mathcal{L}-1/2)}(y,x)E(s,\tau,y,x)\frac{\partial I}{\partial x}(x,\tau)$
$=(I)+(II)$ with $(I)=-\int dx\partial_x e^{(s-\tau)(\mathcal{L}-1/2)}(y,x)E(s,\tau,y,x)I(x,\tau)$ and $(II)=-\int dxe^{(s-\tau)(\mathcal{L}-1/2)}(y,x)\partial_x E(s,\tau,y,x)I(x,\tau)$. Let us first bound (I). From (128), $(I)=$

$\int dx \frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \frac{2(x-ye^{-(s-\tau)/2})}{4\pi(1-e^{-(s-\tau)})} \exp\left(-\frac{(ye^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right) E(s,\tau,y,x) I(x,\tau)$.
If $s_0 \geq \rho^*$, then $0 \leq E(s,\tau,y,x) \leq C$ (use for this $V(x,\tau) \leq C\tau^{-1}$ which is a consequence of (54), (126), $d\mu_{yx}^{s-\tau}$ is a probability and $s \leq \sigma+\rho \leq \sigma+\rho^* \leq \sigma+s_0 \leq 2\sigma \leq 2\tau$). Using (135), we get
$|(I)| \leq C(K_0,A,C_0') \frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \int \frac{dx}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \times \frac{2|ye^{-(s-\tau)/2}-x|}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \times$
$\exp\left(-\frac{(ye^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right) \left(\tau^{-2\bar{p}}(\log\tau)^{\bar{p}}(1+|x|^{3\bar{p}}) + \tau^{-5/2}\log\tau(1+|x|^3)\right)$
where $\bar{p} = \min(p,2) > 1$. With the change of variables $\xi = \frac{x-ye^{-(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}}$,
$|(I)| \leq C(K_0,A,C_0') \frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \left\{\tau^{-2\bar{p}}(\log\tau)^{\bar{p}} \times\right.$
$\int d\xi|\xi|e^{-\xi^2}(1+|\xi\sqrt{4\pi(1-e^{-(s-\tau)})} - ye^{-(s-\tau)/2}|^{3\bar{p}}) + \tau^{-5/2}\log\tau \times$
$\left.\int d\xi|\xi|e^{-\xi^2}(1+|\xi\sqrt{4\pi(1-e^{-(s-\tau)})} - ye^{-(s-\tau)/2}|^3)\right\}$, hence $|(I)| \leq$

$$(136) C(K_0,A,C_0') \frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \left\{\frac{(\log\tau)^{\bar{p}}}{\tau^{2\bar{p}}}(1+|y|^{3\bar{p}}) + \frac{\log\tau}{\tau^{5/2}}(1+|y|^3)\right\}.$$

Let us bound (II) now. Using the integration by parts formula for Gaussian measures (see [11]), we have $\partial_x E(s,\sigma,y,x)$:

$$= \frac{1}{2}\int_0^{s-\tau}\int_0^{s-\tau} d\tau_1 d\tau_2 \partial_x\Gamma(\tau_1,\tau_2) \int d\mu_{yx}^{s-\tau}(\omega) V'(\omega(\tau_1),\sigma+\tau_1) \times$$

$$V'(\omega(\tau_2),\sigma+\tau_2) e^{\int_0^{s-\tau} d\tau_3 V(\omega(\tau_3),\sigma+\tau_3)} \tag{137}$$

$$+\frac{1}{2}\int_0^{s-\tau} d\tau_1 \partial_x\Gamma(\tau_1,\tau_1) \int d\mu_{yx}^{s-\tau}(\omega) V''(\omega(\tau_1),\sigma+\tau_1) e^{\int_0^{s-\tau} d\tau_3 V(\omega(\tau_3),\sigma+\tau_3)}.$$

By (54), we have $|\frac{\partial^n V}{\partial y^n}| \leq Cs^{-n/2}$ for $n=0,1,2$. Combining this with (127) and (137), we get (for $s_0 \geq \rho^*$)
$|\partial_x E(s,\sigma,y,x)| \leq Cs^{-1}(s-\tau)(1+s-\tau)(|y|+|x|)$.
Using this, (128) and (135), we obtain
$|(II)| \leq e^{(s-\tau)/2} \int \frac{dx}{\sqrt{4\pi(1-e^{-(s-\tau)})}} \exp\left(-\frac{(y^{-(s-\tau)/2}-x)^2}{4\pi(1-e^{-(s-\tau)})}\right)(|y|+|x|) \times$
$Cs^{-1}(s-\tau)(1+s-\tau)C(K_0,A,C_0')\left\{\tau^{-2\bar{p}}(\log\tau)^{\bar{p}}(1+|x|^{3\bar{p}})\right.$
$\left.+\tau^{-5/2}\log\tau(1+|x|^3)\right\}$.
Arguing as for (I), we get:

$$|(II)| \leq C(K_0,A,C_0')e^{(s-\tau)/2}(s-\tau)(1+s-\tau)s^{-1}(1+|y|) \times$$

$$\left\{\frac{(\log\tau)^{\bar{p}}}{\tau^{2\bar{p}}}(1+|y|^{3\bar{p}}) + \frac{\log\tau}{\tau^{5/2}}(1+|y|^3)\right\}. \tag{138}$$

Combining (136) and (138), we obtain
$|\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \leq C(K_0,A,C_0')\left\{s^{-2\bar{p}}(\log s)^{\bar{p}}(1+|y|^{3\bar{p}})\right.$
$\left.+s^{-5/2}\log s(1+|y|^3)\right\} \times$
$\int_\sigma^s \left\{\frac{e^{(s-\tau)/2}}{\sqrt{4\pi(1-e^{-(s-\tau)/2})}} + e^{(s-\tau)/2}(s-\tau)(1+s-\tau)s^{-1}(1+|y|)\right\} d\tau$
$\leq C(K_0,A,C_0')\left\{s^{-2\bar{p}}(\log s)^{\bar{p}}(1+|y|^{3\bar{p}}) + s^{-5/2}\log s\right\} \times$
$(e^{s-\sigma}\sqrt{s-\sigma} + e^{(s-\sigma)/2}((s-\sigma)^2+(s-\sigma)^3)s^{-1}(1+|y|))$ ($s_0 \geq \rho^*$, which implies

$2\tau \geq s$). Multiplying this by $\chi(y,s)$ and replacing some $|y|$ by $2K_0\sqrt{s}$, we get: $\forall s \in [\sigma, \sigma+\rho]$,
$|\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \leq C(K_0, A, C_0')\left\{s^{-(\bar{p}+3)/2} + s^{-5/2}\right\}(1+|y|^3)\times$
$\sqrt{s-\tau}(e^{\rho^*} + e^{\rho^*/2}(\rho^{*3/2} + \rho^{*5/2})s^{-1/2})$. If $s \geq s_0 \geq s_{22}(A, \rho^*)$, then
$|\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau)| \leq Cs^{-2}\sqrt{s-\tau}(1+|y|^3)$ (use $\bar{p} > 1$). Therefore,
$|P_\perp(\chi(y,s)\int_\sigma^s d\tau K_1(s,\tau)\frac{\partial I}{\partial y}(\tau))| \leq Cs^{-2}\sqrt{s-\tau}(1+|y|^3)$.
This concludes the proof of (79).

By definition, $R_1(x,\tau) = \frac{\partial R}{\partial y}(x,\tau) + \frac{\partial V}{\partial y}q(x,\tau)$. From (54), we have
$|\frac{\partial V}{\partial y}(x,\tau)| = 2pb\varphi(x,\tau)^{p-2}(p-1+bx^2/\tau)^{-p/(p-1)}x\tau^{-1}$ with
$b = (p-1)^2/(4(p-a))$. Setting $z = x\tau^{-1/2}$, we easily see that
$|\frac{\partial V}{\partial y}(x,\tau)| \leq C\tau^{-1/2}$. Using lemmas B.1 and B.5, we get
$|R_1(x,\tau)| \leq C\tau^{-(1+\bar{p})}(|x|+|x|^3) + CA^2\tau^{-5/2}\log\tau(1+|x|^3)$
$\leq C\tau^{-(2+\epsilon_2(p))}(1+|x|^3)$ with $\epsilon_2(p) > 0$ if $s_0 \geq s_{33}(A)$. Therefore,
$|K_1(s,\tau)R_1(\tau)| = |\int K_1(s,\tau,y,x)R_1(x,\tau)dx|$
$\leq C\tau^{-(2+\epsilon_2(p))}\int |K_1(s,\tau,y,x)(1+|x|^3)dx$
$\leq C\tau^{-(2+\epsilon_2(p))}e^{(s-\tau)/2}(1+|y|^3)$ by lemma B.2 *i*). Hence,
$|\int_s^\sigma d\tau K_1(s,\tau)R_1(\tau)| \leq C(1+|y|^3)\int_\sigma^s d\tau\tau^{-(2+\epsilon_2(p))}e^{(s-\tau)/2}$
$\leq C(s-\sigma)e^{(s-\sigma)/2}s^{-(2+\epsilon_2(p))}(1+|y|^3)$ if $\sigma \geq s_0 \geq \rho^*$.
Now, if $\sigma \geq s_0 \geq s_{23}(\rho^*)$, then
$|\int_s^\sigma d\tau K_1(s,\tau)R_1(\tau)| \leq C(s-\sigma)e^{\rho^*/2}s^{-(2+\epsilon_2(p))}(1+|y|^3)$
$\leq (s-\sigma)s^{-2}(1+|y|^3)$. By classical arguments, this yields (80).

From lemmas B.2 and B.6, and the fact that $\left|\frac{\partial\varphi}{\partial y}\right| \leq C\tau^{-1/2}$, we have:
$|e^{-\tau}(\frac{\partial\varphi}{\partial y} + r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q)| \leq C(K_0, C_0')A^2\tau^{-1/2}e^{-\tau}$. Therefore, *i*) of lemma B.2 yields:
$K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y} + r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))| \leq C(K_0, C_0')A^2e^{\frac{s-\tau}{2}}\tau^{-1/2}e^{-\tau}$. Hence,
$|\int_\sigma^s d\tau K_1(s,\tau)e^{-\tau}(\frac{\partial\varphi}{\partial y} + r)f_1'(e^{\frac{\tau}{p-1}}(\varphi+q))| \leq C(K_0, C_0')A^2(s-\sigma)e^{\frac{s-\sigma}{2}}\frac{e^{-\sigma}}{\sqrt{\sigma}}$
$\leq C(K_0, C_0')A^2(s-\sigma)e^{\rho^*}s^{-1/2}e^{-\frac{s}{2}}$ if $s_0 \geq \rho^*$
$(s-\sigma)s^{-2}$ if $s \geq s_{24}(K_0, A, \rho^*)$. Thus, by classical arguments, (81) follows.

Since $r_m(s_0) = O(e^{-s_0}) + (m+1)q_{m+1}(s_0)$, we have from lemma 3.1 $|r_0(s_0)| \leq CAs_0^{-2}$, $|r_1(s_0)| \leq Cs_0^{-2}\log s_0$, $|r_\perp(y,s_0)| \leq s_0^{-2}(1+|y|^3)$ and $|r_e(y,s_0)| \leq s_0^{-1/2}$. Applying lemma *iii*) of B.2 with $A' = CA$, $A'' = C$, $A''' = A'''' = 1$ yields (82).

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Deuxième partie

Estimations générales des solutions positives explosives de l'équation de la chaleur non linéaire et notions de profil à l'explosion

Chapitre 1

Estimations uniformes à l'explosion pour les équations de la chaleur non linéaires et applications

Estimations uniformes à l'explosion pour les équations de la chaleur non linéaires et applications†

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On s'intéresse à l'équation de la chaleur non linéaire

$$\text{(1)} \qquad \begin{cases} u_t &= \Delta u + u^p \\ u(0) &= u_0 \geq 0, \end{cases}$$

où u est définie pour $(x,t) \in \mathbb{R}^N \times [0,T)$, $1 < p$ et $(N-2)p < N+2$. Différentes généralisations de cette équation peuvent être considérées (voir [12] pour plus de détails):

$$\text{(2)} \qquad \begin{cases} u_t &= \nabla.(a(x)\nabla u) + b(x)u^p \\ u(0) &= u_0 \geq 0, \end{cases}$$

où u est définie pour $(x,t) \in \Omega \times [0,T)$, $1 < p$ et $(N-2)p < N+2$, $\Omega = \mathbb{R}^N$ ou Ω est un ouvert convexe borné et régulier, $a(x)$ est une matrice symétrique et uniformément elliptique, $a(x)$ et $b(x)$ sont $\mathcal{C}^2$ et bornées.

Plus précisément, on s'intéresse au phénomène d'explosion en temps fini. Une littérature importante est considérée à ce sujet. On pourra citer les travaux de Ball [1], Bricmont et Kupiainen et Lin [3] [2], Chen et Matano [4], Galaktionov et Vazquez [6], Giga et Kohn [7] [8] [9], Herrero et Velazquez [10] [11] (voir [12] et [13] pour les références). Dans la suite, on note T le temps d'explosion de $u(t)$, une solution explosive de (1).

Le problème qui nous intéresse est celui d'obtenir des estimations uniformes optimales et de donner des applications de telles estimations.

Pour de telles estimations, on est amené à considérer l'équation (1) dans sa forme auto-similaire: pour tout $a \in \mathbb{R}^N$, on pose

$$\text{(3)} \qquad \begin{aligned} y &= \tfrac{x-a}{\sqrt{T-t}} \\ s &= -\log(T-t) \\ w_a(y,s) &= (T-t)^{\frac{1}{p-1}} u(x,t). \end{aligned}$$

On a alors que $w_a = w$ satisfait $\forall s \geq -\log T$, $\forall y \in \mathbb{R}^N$:

$$\text{(4)} \qquad \frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + w^p.$$

Le problème est d'estimer $w_a(s)$ quand $s \to +\infty$, que a soit un point régulier ou un point d'explosion (a est dit point d'explosion lorsqu'il existe $(a_n, t_n) \to (a,T)$ tel que $u(a_n,t_n) \to +\infty$) de faCcon uniforme.

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† Note parue dans les actes du séminaire EDP 1996-1997, École Polytechnique, pp. XIX-1 XIX-8.

Giga et Kohn ont démontré qu'en fait les variables auto-similaires sont les bonnes variables pour mesurer les solutions explosives dans les sens suivant: il existe $\epsilon_0 > 0$ tel que $\forall s \geq s_0^*$,

$$\epsilon_0 \leq |w(s)|_{L^\infty} \leq \frac{1}{\epsilon_0}.$$

On se propose dans un premier temps d'affiner ce résultat pour obtenir de la compacité dans le problème.

1 Un théorème de Liouville pour l'équation (4)

Pour ceci, on s'intéresse à un problème de classification de solutions globales. On a le résultat suivant:

Théorème 1 (Théorème de Liouville pour (4)) *Soit w une solution de (4) définie pour $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ telle que $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $0 \leq w(y,s) \leq C$. Alors, on est nécessairement dans l'un des cas suivants:*
i) $w \equiv 0$,
ii) $w \equiv \kappa$ où $\kappa = (p-1)^{-\frac{1}{p-1}}$,
iii) $\exists s_0 \in \mathbb{R}$ tel que $w(y,s) = \varphi(s - s_0)$ où

$$\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}.$$

Remarque: Remarquons que φ est une connexion dans L^∞ des deux points critiques de (4): 0 et κ. En effet,

$$\dot{\varphi} = -\frac{\varphi}{p-1} + \varphi^p,\ \varphi(-\infty) = \kappa,\ \varphi(+\infty) = 0.$$

Remarque: Il suffit d'avoir une solution de (4) définie sur $(-\infty, s^*)$ pour avoir un théorème de classification (voir [12]).

On peut obtenir comme corollaire

Corollaire 1 *Soit u une solution de (1) définie pour $(x,t) \in \mathbb{R}^N \times (-\infty, 0)$ telle que $\forall (x,t) \in \mathbb{R}^N \times (-\infty, 0)$, $0 \leq u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}$. Alors,*
soit $u \equiv 0$,
soit $\exists T^* \geq 0$ tel que $u(x,t) = \kappa(T^* - t)^{-\frac{1}{p-1}}$.

Pour les démonstrations, voir [12]. Les outils clefs de la démonstration sont:
i) une classification des comportements linéaires de $w(s)$ quand $s \to -\infty$ dans $L^2_\rho(\mathbb{R}^N)$ $(L^\infty_{loc}(\mathbb{R}^N))$ où $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}$,
ii) les transformations géométriques

$$w(y,s) \to w_{a,b}(y,s) = w(y + ae^{\frac{s}{2}}, s + b)$$

pour $a \in \mathbb{R}^N$ et $b \in \mathbb{R}$,
iii) un critère d'explosion en temps fini dans les variables auto-similaires: si pour un certain $s_0 \in \mathbb{R}$, $\int w(y,s_0)\rho(y)dy > \int \kappa\rho(y)dy$, alors $w(s)$ explose en temps fini.

2 Estimations optimales à l'explosion

Par un argument de compacité, on obtient les estimations uniformes suivantes sur la solution $w(s)$ de (4):

Théorème 2 (Estimations optimales à l'ordre zéro sur $w(s)$)
Si $w(s_0) \in H^1(\mathbb{R}^N)$, alors
$\|w(s)\|_{L^\infty(\mathbb{R}^N)} \to \kappa$ *et* $\|\nabla w(s)\|_{L^\infty(\mathbb{R}^N)} + \|\Delta w(s)\|_{L^\infty(\mathbb{R}^N)} \to 0$ *quand* $s \to +\infty$.

Remarque: Cette estimation est aussi valable pour un ensemble de solutions (voir [12]).

Cette estimation est très importante car elle donne pour une solution la convergence de $w_a(s)$ vers un ensemble limite dans L^∞_{loc} uniformément par rapport à $a \in \mathbb{R}^N$. Ceci nous permet ensuite par linéarisation autour de cet ensemble de démontrer le

Théorème 3 (Estimation optimale à l'ordre un sur $w(s)$) *Sous les hypothèses du Théorème 2, $\forall \epsilon_0 > 0$, il existe $s(\epsilon_0)$ tel que $\forall s \geq s(\epsilon_0)$, $\exists C_1, C_2 > 0$ tels que*

$$\begin{aligned}
\|w(s)\|_{L^\infty} &\leq \kappa + (\frac{N\kappa}{2p} + \epsilon_0)\frac{1}{s} \\
\|\nabla w(s)\|_{L^\infty} &\leq \frac{C_1}{\sqrt{s}} \\
\|\nabla^2 w(s)\|_{L^\infty} &\leq \frac{C_2}{s}.
\end{aligned}$$

Remarque: Dans le cas $N = 1$, en utilisant une propriété de Sturme Développée par Chen et Matano (qui affirme que le nombre d'oscillations en espace de la solution est une fonction décroissante du temps), Herrero et Velazquez (et Filippas et Kohn) ont montré des estimations de ce type.
Remarque: La constante $\frac{N\kappa}{2p}$ est optimale (voir Herrero et Velazquez, Bricmont et Kupiainen, Merle et Zaag).

3 Localisation à l'explosion

Le Théorème 2 implique que dans la zone singulière du type $\{y \mid w(y,s) \geq \frac{\kappa}{2}\}$, Δw est petit devant w^p (ou de faCcon équivalente, Δu est petit devant u^p). Un phénomène de localisation sous critique introduit par Zaag [15] (sous le seuil de la constante) nous permet de propager ces estimations dans les zones singulières: "$u(x,t)$ grand". Il en découle le théorème suivant:

Théorème 4 (Comparaison avec l'équation différentielle ordinaire) *Si $u_0 \in H^1(\mathbb{R}^N)$, alors $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ tel que $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \mathbb{R}^N$,*

$$|u_t - u^p| \leq \epsilon u^p + C_\epsilon.$$

Remarque: Ainsi, on démontre que la solution de l'équation aux dérivées partielles est comparable uniformément et globalement en espace-temps à une équation différentielle ordinaire (localisée par définition).

On peut noter que le résultat reste vrai pour une suite de solutions sous certaines conditions.

Remarque: De multiples corollaires découlent de ce théorème. Par exemple, $\forall \epsilon_0 > 0$, il existe $t_0(\epsilon_0) < T$ tel que pour tout $a \in \mathbb{R}^N$, $t \in [t_0, T)$, si $u(a,t) \leq (1-\epsilon_0)\kappa(T-t)^{-\frac{1}{p-1}}$, alors, a n'est pas point d'explosion. (Ceci précise un résultat de Giga et Kohn où $t_0 = t_0(\epsilon_0, a)$.

4 Notion de Profil au voisinage d'un point d'explosion

On considère maintenant $a \in \mathbb{R}^N$ un point d'explosion de $u(t)$ solution de (1). Par invariance par translation, on se ramène à $a = 0$. La question est de savoir si $u(t)$ (ou $w_0(s)$ définie en (3)) a un comportement universel ou pas quand $t \to T$ (ou $s \to +\infty$).

Filippas, Kohn, Liu, Herrero et Velazquez ont démontré que w évoluait suivant l'une des deux possibilités suivantes:

- $\forall R > 0, \ \sup_{|y| \leq R} \left| w(y,s) - \left[\kappa + \frac{\kappa}{2ps} \left(trA_k - \frac{1}{2} y^T A_k y \right) \right] \right| = O\left(\frac{1}{s^{1+\delta}} \right)$

quand $s \to +\infty$ pour un certain $\delta > 0$ avec

$$A_k = Q \begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1},$$

$k \in \{0, 1, ..., N-1\}$, Q une matrice $N \times N$ orthogonale et I_{N-k} l'identité des matrices $(N-k) \times (N-k)$.

- $\forall R > 0, \ \sup_{|y| \leq R} |w(y,s) - \kappa| \leq C(R) e^{-\epsilon_0 s}$ pour un certain $\epsilon_0 > 0$.

Dans un certain sens, ces résultats démarquent mal d'un point de vue physique la transition entre les zones singulière ($w \geq \alpha$ où $\alpha > 0$) et régulière ($w \simeq 0$). En utilisant la théorie de la renormalisation, Bricmont et Kupiainen ont démontré dans [3] l'existence d'une solution de (4) telle que

$$\forall s \geq s_0, \ \forall y \in \mathbb{R}^N, |w(y,s) - f_0(\frac{y}{\sqrt{s}})| \leq \frac{C}{\sqrt{s}}$$

où $f_0(z) = (p - 1 + \frac{(p-1)^2}{4p} |z|^2)^{-\frac{1}{p-1}}$. Merle et Zaag ont démontré dans [14] le même résultat grâce à des techniques de réduction en dimension finie. Ils y démontrent aussi la stabilité par rapport aux données initiales de telles comportements.

Dans [15], Zaag montre que dans ce cas, $u(x,t) \to u^*(x)$ quand $t \to T$ uniformément sur $\mathbb{R}^N \backslash \{0\}$ et que $u^*(x) \sim \left[\frac{8p|\log|x||}{(p-1)^2|x|^2} \right]^{\frac{1}{p-1}}$ quand $x \to 0$.

Dans un premier temps, on est en mesure de démontrer grâce aux estimations du Théorème 4, un théorème de classification des profils dans la variable $\frac{y}{\sqrt{s}}$ (qui sépare partie singulière et régulière dans le cas non dégénéré).

Théorème 5 (Classification des profils à l'explosion)
Il existe $k \in \{0, 1, ..., N-1\}$ et une matrice $N \times N$ orthogonale Q tels que

$w(Q(z)\sqrt{s}, s) \to f_k(z)$ *uniformément sur tout compact* $|z| \le C$, *où*
$f_k(z) = (p-1+\frac{(p-1)^2}{4p}\sum_{i=1}^{N-k}|z_i|^2)^{-\frac{1}{p-1}}$ *si* $k \le N-1$ *et* $f_N(z) = \kappa = (p-1)^{-\frac{1}{p-1}}$.

Un des problèmes intéressants qui en découle est de relier toutes les notions de profils connues: profil pour $|y|$ borné, $\frac{|y|}{\sqrt{s}}$ borné ou $x \simeq 0$. On démontre que ces notions sont équivalentes dans le cas d'une solution qui explose en un point de faCcon non dégénérée (cas générique), ce qui répond à de nombreuses questions posées dans des travaux précédents.

Théorème 6 (Équivalence des comportements explosifs en un point)
Soit a un point d'explosion isolé de $u(t)$ *solution de (1). On a l'équivalence des trois comportements suivants de* $u(t)$ *et de* $w_a(s)$ *(définie en (3)):*

i) $\forall R > 0$, $\sup_{|y|\le R}\left|w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ *quand* $s \to +\infty$,

ii) $\forall R > 0$, $\sup_{|z|\le R}\left|w(z\sqrt{s}, s) - f_0(z)\right| \to 0$ *quand* $s \to +\infty$ *avec* $f_0(z) = (p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

iii) $\exists \epsilon_0 > 0$ *tel que pour tout* $|x-a| \le \epsilon_0$, $u(x,t) \to u^*(x)$ *quand* $t \to T$ *et* $u^*(x) \sim \left[\frac{8p|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1}{p-1}}$ *quand* $x \to a$.

Remarque: . Dans le cas $N = 1$, certaines implications étaient déjà démontrées.

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Chapitre 2

Optimal estimates for blow-up rate and behavior for nonlinear heat equations

Optimal estimates for blow-up rate and behavior for nonlinear heat equations†

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Abstract: We first describe all positive bounded solutions of

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + w^p.$$

where $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, $1 < p$ and $(N-2)p \leq N+2$. We then obtain for blow-up solutions $u(t)$ of

$$\frac{\partial u}{\partial t} = \Delta u + u^p$$

uniform estimates at the blow-up time and uniform space-time comparison with solutions of $u' = u^p$.

1 Introduction

We consider the following nonlinear heat equation:

$$(1) \qquad \begin{array}{rcll} \frac{\partial u}{\partial t} & = & \Delta u + |u|^{p-1} u & \text{in} \quad \Omega \times [0,T) \\ u & = & 0 & \text{on} \quad \partial\Omega \times [0,T) \end{array}$$

where $u(t) \in H^1(\Omega)$ and $\Omega = \mathbb{R}^N$ (or Ω is a convex domain).

We assume in addition that

$$1 < p, \ (N-2)p < N+2 \text{ and } u(0) \geq 0.$$

In this paper, we are interested in blow-up solutions $u(t)$ of equation (1): $u(t)$ blows-up in finite time T if u exists for $t \in [0,T)$ and $\lim\limits_{t \to T} \|u(t)\|_{H^1} = +\infty$. In this case, one can show that u has at least one blow-up point, that is $a \in \Omega$ such that there exists $(a_n, t_n)_{n \in \mathbb{N}}$ satisfying $(a_n, t_n) \to (a, T)$ and $|u(a_n, t_n)| \to +\infty$. We aim in this work at studying the blow-up behavior of $u(t)$. In particular, we are interested in obtaining uniform estimates on $u(t)$ at or near the singularity, that is estimates "basically" independent of initial data.

We will give two types of uniform estimates: the first one holds especially at the singular set (Theorem 1) and the other one consists in surprising global estimates in space and time (Theorem 3). It will be deduced from the former by some strong control of the interaction between regular and singular parts of the solution. Various applications of this type of estimates will be given in [12].

For the first type of estimates, we introduce for each $a \in \Omega$ (a may be a blow-up point of u or not) the following similarity variables:

$$(2) \qquad \begin{array}{rcl} y & = & \frac{x-a}{\sqrt{T-t}} \\ s & = & -\log(T-t) \\ w_a(y,s) & = & (T-t)^{\frac{1}{p-1}} u(x,t). \end{array}$$

† Article paru dans Comm. Pure Appl. Math. 51, 1998, pp. 139-196.

w_a $(= w)$ satisfies $\forall s \geq -\log T$, $\forall y \in D_{a,s}$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w \tag{3}$$

where

$$D_{a,s} = \{y \in \mathbb{R}^N \mid a + ye^{-s/2} \in \Omega\}. \tag{4}$$

We introduce also the following Lyapunov functional:

$$E(w) = \frac{1}{2} \int |\nabla w|^2 \rho dy + \frac{1}{2(p-1)} \int |w|^2 \rho dy - \frac{1}{p+1} \int |w|^{p+1} \rho dy \tag{5}$$

$$\text{where } \rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{N/2}} \tag{6}$$

and the integration is done over the definition set of w.

The study of $u(t)$ near (a, T) where a is a blow-up point is equivalent to the study of the long time behavior of w_a. Note that $D_{a,s} \neq \mathbb{R}^N$ in the case $\Omega \neq \mathbb{R}^N$. This in fact is not a problem since we know from [8] that $a \notin \partial\Omega$ in the case Ω is $C^{2,\alpha}$, and therefore, for a given $a \in \Omega$, $D_{a,s} \to \mathbb{R}^N$ as $s \to +\infty$. Let $a \in \Omega$ be a blow-up point of u.

If Ω is a bounded convex domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, then Giga and Kohn prove in [7] that:

$$\begin{array}{rrcl} \forall s \geq -\log T, & \|w_a(y,s)\|_{L^\infty(D_{a,s})} & \leq & C \text{ or equivalently} \\ \forall t \in [0,T), & \|u(x,t)\|_{L^\infty(\Omega)} & \leq & C(T-t)^{-\frac{1}{p-1}}. \end{array} \tag{7}$$

They also prove in [7] and [8] (see also [6]) that for a given blow-up point $a \in \Omega$,

$$\lim_{s \to +\infty} w_a(y,s) = \lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) = \kappa$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$, uniformly on compact subsets of $\mathbb{R}^N$. The result is pointwise in a. Besides, for a.e y, $\lim_{s \to +\infty} \nabla w_a(y,s) = 0$.

Let us denote $L^\infty(D_{a,s})$ by L^∞.

In this paper, we first obtain uniform (on a and in some sense on $u(0)$) sharp estimates on w_a, and we find a precise long time behavior for $\|w_a(s)\|_{L^\infty}$, $\|\nabla w_a(s)\|_{L^\infty}$ and $\|\Delta w_a(s)\|_{L^\infty}$ (global estimates).

Theorem 1 (Optimal bound on $u(t)$ at blow-up time) *Assume that Ω is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Consider $u(t)$ a blow-up solution of equation (1) which blows-up at time T. Assume in addition $u(0) \geq 0$ and $u(0) \in H^1(\Omega)$. Then*

$$(T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} \to \kappa = (p-1)^{-\frac{1}{p-1}} \text{ as } t \to T$$

and

$$(T-t)^{\frac{1}{p-1}+1} \|\Delta u(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to T,$$

or equivalently for any $a \in \Omega$,

$$\|w_a(s)\|_{L^\infty} \to \kappa \text{ as } s \to +\infty$$

and

$$\|\Delta w_a(s)\|_{L^\infty} + \|\nabla w_a(s)\|_{L^\infty} \to 0 \text{ as } s \to +\infty.$$

Remark: We can point out that we do not consider local norm in w variable such as $L^2(d\mu)$ with $d\mu = e^{-|y|^2/4}dy$ as a center manifold theory for equation (3) would suggest. Instead, we use L^∞ norm which yields results uniform with respect to $a \in \Omega$. Indeed, we have from (2) that $\forall a, b \in \Omega$, $\forall (y,s) \in D_{b,s}$,

$$w_b(y,s) = w_a(y + (b-a)e^{\frac{s}{2}}, s),$$

which yields $\|w_a\|_{L^\infty} = \|w_b\|_{L^\infty}$, $\|\nabla w_a\|_{L^\infty} = \|\nabla w_b\|_{L^\infty}$ and $\|\Delta w_a\|_{L^\infty} = \|\Delta w_b\|_{L^\infty}$).

One interest of Theorem 1 is that in fact, its proof yields the following compactness result:

Theorem 1' (Compactness of blow-up solutions of (1)) *Assume that Ω is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Consider $(u_n)_{n\in\mathbb{N}}$ a sequence of nonnegative solutions of equation (1) such that for some $T > 0$ and for all $n \in \mathbb{N}$, u_n is defined on $[0,T)$ and blows-up at time T. Assume also that $\|u_n(0)\|_{H^2(\Omega)}$ is bounded uniformly in n. Then*

$$\sup_{n\in\mathbb{N}} (T-t)^{\frac{1}{p-1}} \|u_n(t)\|_{L^\infty(\Omega)} \to \kappa \text{ as } t \to T$$

and

$$\sup_{n\in\mathbb{N}} \left((T-t)^{\frac{1}{p-1}+1} \|\Delta u_n(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u_n(t)\|_{L^\infty(\Omega)} \right) \to 0$$

as $t \to T$.

Remark: The same results can be proved for the following heat equation:

$$\frac{\partial u}{\partial t} = \nabla.(a(x)\nabla u) + b(x)f(u), \;\; u(0) \geq 0$$

where $f(u) \sim u^p$ as $u \to +\infty$, $(a(x))$ is a symmetric, bounded and uniformly elliptic matrix, $b(x)$ is bounded, and $a(x)$ and $b(x)$ are C^1.

Let us point out that this result is optimal. One way to see it is by the following Corollary which improves the local lower bound on the blow-up solution given in [8] by Giga and Kohn.

Corollary 1 (Lower bound on the blow-up behavior for equation (1)) *Assume that Ω is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Then for all nonnegative solution $u(t)$ of (1) such that $u(0) \in H^1(\Omega)$ and $u(t)$ blows-up at time T, and for all $\epsilon_0 \in (0,1)$, there exists $t_0 = t_0(\epsilon_0, u_0) < T$ such that if for some $a \in \Omega$ and some $t \in [t_0, T)$ we have*

$$0 \leq u(a,t) \leq (1-\epsilon_0)\kappa(T-t)^{-\frac{1}{p-1}}, \tag{8}$$

then a is not a blow-up point of $u(t)$.

Remark: The result is still true for a sequence of nonnegative solutions u_n blowing-up at $T > 0$ and satisfying the assumptions of Theorem 1', with a t_0 independent of n.

Remark: κ is the optimal constant giving such a result. The result of [8] was the same except that $(1-\epsilon_0)\kappa$ was replaced by ϵ_0 small and it was required that

(8) is true for all $(x,t) \in B(a,r) \times [T-r^2,T)$ for some $r>0$ (no sign condition was required there).

The proof of Theorem 1 relies strongly on the characterization of all connections between two critical points of equation (3) in L^∞_{loc}. Due to [6], the only bounded global nonnegative solutions of the stationary problem associated to (3) in $\mathbb{R}^N$ are 0 and κ, provided that $(N-2)p \le N+2$. Here we classify the solutions $w(y,s)$ of (3) defined on $\mathbb{R}^N \times \mathbb{R}$ and connecting two of the cited critical points between them, and we obtain the surprising result:

Theorem 2 (Classification of connections between critical points of (3)) *Assume that $1<p$ and $(N-2)p<N+2$ and that w is a global nonnegative solution of (3) defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ bounded in L^∞. Then necessarily one of the following cases occurs:*

i) $w \equiv 0$ or $w \equiv \kappa$,

or ii) there exists $s_0 \in \mathbb{R}$ such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y,s) = \varphi(s-s_0)$ where

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}. \tag{9}$$

Note that φ is the unique global solution (up to a translation) of

$$\varphi_s = -\frac{\varphi}{p-1} + \varphi^p$$

satisfying $\varphi \to \kappa$ as $s \to -\infty$ and $\varphi \to 0$ as $s \to +\infty$.

Remark: This result is in the same spirit as the result of Berestycki and Nirenberg [1], and Gidas, Ni and Nirenberg [5]. Here, the moving plane technique is not used, even though the proof uses some elementary geometrical transformations. It is unclear whether the result holds without a sign condition or not. The assumption w is bounded in L^∞ and is defined for s up to $+\infty$ is not really needed, in the following sense:

Corollary 2 *Assume that $1<p$ and $(N-2)p<N+2$ and that w a nonnegative solution of (3) defined for $(y,s) \in \mathbb{R}^N \times (-\infty, s^*)$ where s^* is finite or $s^* = +\infty$. Assume in addition that there is a constant C_0 such that $\forall a \in \mathbb{R}^N$, $\forall s \le s^*$, $E_a(w(s)) \le C_0$, where*

$$E_a(w(s)) = E(w(.+ae^{\frac{s}{2}}, s)) \tag{10}$$

and E is defined in (5). Then, one of the following cases occurs:

i) $w \equiv 0$ or $w \equiv \kappa$,

or ii) $\exists s_0 \in \mathbb{R}$ such that $\forall (y,s) \in \mathbb{R}^N \times (-\infty, s^*)$, $w(y,s) = \varphi(s-s_0)$ where

$$\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}},$$

or iii) $\exists s_0 \ge s^*$ such that $\forall (y,s) \in \mathbb{R}^N \times (-\infty, s^*)$, $w(y,s) = \psi(s-s_0)$ where

$$\psi(s) = \kappa(1-e^s)^{-\frac{1}{p-1}}.$$

Theorem 2 has an equivalent formulation for solutions of (1):

Corollary 3 (A Liouville theorem for equation (1)) *Assume that $1<p$ and $(N-2)p<N+2$ and that u is a nonnegative solution in L^∞ of (1) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. Assume in addition that $0 \le u(x,t) \le C(T-t)^{-\frac{1}{p-1}}$. Then $u \equiv 0$ or there exist $T_0 \ge T$ such that $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \kappa(T_0-t)^{-\frac{1}{p-1}}$.*

Remark: $u \equiv 0$ or u blows-up in finite time $T_0 \geq T$.

The third main result of the paper shows that near blow-up time, the solutions of equation (1) behave globally in space like the solutions of the associated ODE:

Theorem 3 *Assume that Ω is a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Consider $u(t)$ a nonnegative solution of equation (1) which blows-up at time $T > 0$. Assume in addition that $u(0) \in H^1(\mathbb{R}^N)$ if $\Omega = \mathbb{R}^N$. Then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \Omega$,*

$$|\frac{\partial u}{\partial t} - |u|^{p-1}u| \leq \epsilon |u|^p + C_\epsilon. \tag{11}$$

Remark: (11) is true until the blow-up time. Let us point out that the result is global in time and in space. The same result holds for a sequence u_n as before (Theorem 1'). For clear reasons, the result is optimal.
Remark: Let us note that the result is still true for equation

$$\frac{\partial u}{\partial t} = \nabla.(a(x)\nabla u) + b(x)f(u)$$

where $f(u) \sim u^p$ as $u \to +\infty$, $(a(x))$ is a symmetric, bounded and uniformly elliptic matrix, $b(x)$ is bounded, and $a(x)$ and $b(x)$ are C^1.
The conclusion in this case is

$$|\frac{\partial u}{\partial t} - b(x)f(u)| \leq \epsilon |f(u)| + C_\epsilon.$$

It is unclear whether Theorems 1, 2 and 3 hold without a sign condition.
Remark: $u' = u^p$ is a reversible equation. Therefore the non reversible equation behaves like a reversible equation near and at the blow-up time. Theorem 3 localizes the equation. In particular, it shows that the interactions between two singularities or one singularity and the "regular" region are bounded up to the blow-up time.

Note that Theorem 3 has obvious corollaries. For example:
If x_0 is a blow-up point, then

- $u(x,t) \to +\infty$ as $(x,t) \to (x_0, T)$ (In other words, u is a continuous function in $\bar{\mathbb{R}}$ of $(x,t) \in \Omega \times (0,T)$).

- $\exists \epsilon_0 > 0$ such that for all $x \in B(x_0, \epsilon_0)$ and $t \in (T - \epsilon_0, T)$, we have $\frac{\partial u}{\partial t}(x,t) > 0$.

Let us notice that theorems 1 and 3 have interesting applications in the understanding of the asymptotic behavior of blow-up solutions $u(t)$ of (1) near a given blow-up point x_0. Various points of view has been adopted in the literature ([8], [2], [9], [14]) to describe this behavior. In [12], we sharpen these estimates and put them in a relation.

In the second section, we see how Theorems 1 and 3 are proved using Theorem 2. The third section is devoted to the proof of Theorem 2.

2 Optimal blow-up estimates for equation (1)

In this section, we assume that Theorem 2 holds and prove Theorems 1 and 1', Corollary 1 and Theorem 3. The first three are mainly a consequence of compactness procedure and Theorem 2. Theorem 3 follows from Theorem 1 and scaling properties of equation (1) used in a suitable way.

2.1 L^∞ estimates for the solution of (1)

We prove Theorems 1 and 1' and Corollary 1 in this subsection.

Proof of Theorem 1: Let $u(t)$ be a nonnegative solution of equation (1) defined on $[0,T)$, which blows-up at time T and satisfies $u(0) \in H^1(\Omega)$. It is clear that the estimates on w_a for all $a \in \Omega$ follow from the estimates on u by (2). In addition, the estimates on u follow from the estimates on w_a for a particular $a \in \Omega$ still by (2). Hence, we consider $a \in \Omega$ a blow-up point of u and prove the estimates on this particular w_a defined by

$$w_a(y,s) = e^{-\frac{s}{p-1}} u(a + ye^{-\frac{s}{2}}, T - e^{-s}).$$

Note that we have $\forall a, b \in \Omega$, $\forall (y,s) \in D_{b,s}$,

$$w_b(y,s) = w_a(y + (b-a)e^{\frac{s}{2}}, s).$$

We proceed in three steps: in a first step, we show that w_a, ∇w_a and Δw_a are uniformly bounded (without any precision on the bounds). Then, we show in Step 2 that blow-up for equation (1) must occur inside a compact set $K \subset \Omega$ and that u, ∇u and Δu are bounded in $\Omega \backslash K$. We finally find the optimal bounds on w_a through a contradiction argument.

Let us recall the expression of the energy $E(w)$ introduced in (5), since it will be useful for further estimates:

$$(12)\ E(w_a) = \frac{1}{2}\int |\nabla w_a|^2 \rho dy + \frac{1}{2(p-1)}\int |w_a|^2 \rho dy - \frac{1}{p+1}\int |w_a|^{p+1} \rho dy$$

where ρ is defined in (6) and integration is done over the definition set of w. By means of the transformation (2), (12) yields a local energy for equation (1):

$$\begin{aligned} \mathcal{E}_{a,t}(u) &= t^{\frac{2}{p-1}-\frac{N}{2}+1} \int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right] \rho(\frac{x-a}{\sqrt{t}})dx \\ (13) \qquad &+ \frac{1}{2(p-1)} t^{\frac{2}{p-1}-\frac{N}{2}} \int |u(x)|^2 \rho(\frac{x-a}{\sqrt{t}})dx. \end{aligned}$$

Without loss of generality, we can suppose $a = 0$. We recall that the notation L^∞ stands for $L^\infty(D_{0,s})$.

Step 1: Preliminary estimates on w

Lemma 2.1 (Giga-Kohn, Uniform estimates on w) *There exists a positive constant M such that $\forall s \geq -\log T + 1$, $\forall y \in D_{0,s}$,*

$$|w_0(y,s)| + |\nabla w_0(y,s)| + |\Delta w_0(y,s)| + |\nabla \Delta w_0(t,s)| \leq M$$

$$\textit{and } |\frac{\partial w}{\partial s}(y,s)| \leq M(1+|y|).$$

Let us recall the main steps of the proof:
Since $u(0) \geq 0$, we know from Giga and Kohn [8] that there exists $B > 0$ such that

(14) $$\forall t \in [0,T),\ \forall x \in \Omega,\ |u(x,t)| \leq B(T-t)^{-\frac{1}{p-1}}.$$

In order to prove this, they argue by contradiction and construct by scaling properties of equation (3) a solution of

$$\begin{cases} 0 &= \Delta v + v^p \text{ in } \mathbb{R}^N \\ v &\geq 0 \\ v(0) &\geq \frac{1}{2} \end{cases}$$

which does not exist if $(N-2)p < N+2$ and $p > 1$.

The estimate on w_0 is equivalent to (14).

For $s_0 \geq -\log T + 1$ and $y_0 \in D_{0,s_0}$, consider $W(y',s') = w_0(y' + y_0 e^{\frac{s}{2}}, s_0 + s')$. Then $W(0,0) = w_0(y_0,s_0)$ and W satisfies also (3). If $y_0 e^{-\frac{s_0}{2}}$ (which is in Ω) is not near the boundary, then we have $|W(y',s')| \leq M$ for all $(y',s') \in B(0,1) \times [-1,1]$. By parabolic regularity (see lemma 3.3 in [7] for a statement), we obtain $|\nabla W(0,0)| + |\Delta W(0,0)| + |\nabla \Delta W(0,0)| \leq M' = M'(M)$. If $y_0 e^{-\frac{s_0}{2}}$ is near the boundary, then lemma 3.4 in [7] allows to get the same conclusion. Since this is true for all (y_0,s_0), we have the bound for ∇w_0, Δw_0 and $\nabla \Delta w_0$.

The estimate on $\frac{\partial w_0}{\partial s}$ follows then by equation (3).

Step 2: No blow-up for u outside a compact

Proposition 2.1 (Uniform boundedness of $u(x,t)$ outside a compact) *Assume that $\Omega = \mathbb{R}^N$ and $u(0) \in H^1(\mathbb{R}^N)$, or that Ω is a convex bounded $C^{2,\alpha}$ domain. Then there exist $C > 0$, $t_1 < T$ and K a compact set of Ω such that $\forall t \in [t_1,T)$, $\forall x \in \Omega \backslash K$, $|u(x,t)| + |\nabla u(x,t)| + |\Delta u(x,t)| \leq C$.*

Proof: *Case $\Omega = \mathbb{R}^N$ and $u(0) \in H^1(\mathbb{R}^N)$*: Giga and Kohn prove in [8] that uniform estimates on $\mathcal{E}_{a,t}$ (13) give uniform estimates in L^∞_{loc} on the solution of (1). More precisely,

Proposition 2.2 (Giga-Kohn) *Let u be a solution of equation (1).*

i) If for all $x \in B(x_0,\delta)$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$, then $\forall x \in B(x_0,\frac{\delta}{2})$, $\forall t \in (\frac{t_0+T}{2},T)$, $|u(t,x)| \leq \eta(\sigma)(T-t)^{-\frac{1}{p-1}}$ where $\eta(\sigma) \leq c\sigma^\theta$, $\theta > 0$, and c and θ depend only on p.

ii) Assume in addition that $\forall x \in B(x_0,\delta)$, $|u(\frac{t_0+T}{2},x)| \leq M$. There exists $\sigma_0 = \sigma_0(p) > 0$ such that if $\sigma \leq \sigma_0$, then $\forall x \in B(x_0,\frac{\delta}{4})$, $\forall t \in (\frac{t_0+T}{2},T)$, $|u(t,x)| \leq M^*$ where M^* depends only on M, δ, T and t_0.

Proof: see Proposition 3.5 and Theorem 2.1 in [8]. ∎

Now, since $u(0) \in H^1(\mathbb{R}^N)$, we have $u(t) \in H^1(\mathbb{R}^N)$ for all $t \in [0,T)$. Therefore, for fixed t_0 and $\sigma \leq \sigma_0$, (13), (6) and the dominated convergence theorem yield the existence of a compact $K_0 \subset \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N \backslash K_0$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$.
Hence, $ii)$ of Proposition 2.2 applied to $u(.+x_1,.)$ for $x_1 \in K_0$ and with $\delta = 1$, asserts the existence of a compact $K_1 \subset \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N \backslash K_1$, $\forall t \in (\frac{t_0+T}{2},T)$, $|u(x,t)| \leq M^*$.

Parabolic regularity (see lemma 3.3 in [7] for a statement) implies the estimates on ∇u and Δu on $\Omega \backslash K$ with a compact K containing K_1.

Case Ω is a bounded convex $C^{2,\alpha}$ domain: The main feature in the proof of the estimate on $|u(x,t)|$ is the result of Giga and Kohn which asserts that blow-up can not occur at the boundary (Theorem 5.3 in [8]). The bounds on ∇u and Δu follow from a similar argument as before (see lemma 3.4 in [7]).

Step 3: Conclusion of the proof

The result has been proved pointwise. Therefore, the question is in some sense to prove it uniformly.

We want to prove that $\|w_0(s)\|_{L^\infty} \to \kappa$ as $s \to +\infty$.

From [7] and [8], we know that $|w_b(0,s)| \to \kappa$ as $s \to +\infty$ if b is a blow-up point. Since $\|w_0(s)\|_{L^\infty} \geq |w_0(ae^{\frac{s}{2}}, s)| = |w_a(0,s)|$, this implies that

$$\begin{gathered} \liminf_{s\to+\infty} \|w_0(s)\|_{L^\infty} \geq \kappa \\ \text{and } \liminf_{s\to+\infty} \|w_0(s)\|_{L^\infty} + \|\nabla w_0(s)\|_{L^\infty} + \|\Delta w_0(s)\|_{L^\infty} \geq \kappa. \end{gathered} \tag{15}$$

The conclusion will follow if we show that

$$\limsup_{s\to+\infty} \|w_0(s)\|_{L^\infty} + \|\nabla w_0(s)\|_{L^\infty} + \|\Delta w_0(s)\|_{L^\infty} \leq \kappa. \tag{16}$$

Let us argue by contradiction and suppose that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \|w_0(s_n)\|_{L^\infty} + \|\nabla w_0(s_n)\|_{L^\infty} + \|\Delta w_0(s_n)\|_{L^\infty} = \kappa + 3\epsilon_0 \text{ where } \epsilon_0 > 0.$$

We claim that (up to extracting a subsequence), we have

$$\begin{array}{llcl} \text{either} & \lim_{n\to+\infty} \|w_0(s_n)\|_{L^\infty} & = & \kappa + \epsilon_0 \\ \text{or} & \lim_{n\to+\infty} \|\nabla w_0(s_n)\|_{L^\infty} & = & \epsilon_0 \\ \text{or} & \lim_{n\to+\infty} \|\Delta w_0(s_n)\|_{L^\infty} & = & \epsilon_0. \end{array} \tag{17}$$

From Proposition 2.1 and the scaling (2), we deduce for n large enough the existence of $y_n^{(0)}$, $y_n^{(1)}$ and $y_n^{(2)}$ in D_{0,s_n} such that

$$\begin{array}{rcl} \|w_0(s_n)\|_{L^\infty} & = & |w_0(y_n^{(0)}, s_n)|, \\ \text{or } \|\nabla w_0(s_n)\|_{L^\infty} & = & |\nabla w_0(y_n^{(1)}, s_n)|, \\ \text{or } \|\Delta w_0(s_n)\|_{L^\infty} & = & |\Delta w_0(y_n^{(2)}, s_n)|. \end{array} \tag{18}$$

Let $y_n = y_n^{(i)}$ where i is the number of the case which occurs. Since $y_n \in D_{0,s_n}$, (4) implies that $y_n e^{-s_n/2} \in \Omega$. Therefore, we can use (2) and define for each $n \in \mathbb{N}$

$$\begin{array}{rcl} v_n(y,s) & = & w_{y_n e^{-s_n/2}}(y, s+s_n) \\ & = & e^{-\frac{s+s_n}{p-1}} u(ye^{-\frac{s+s_n}{2}} + y_n e^{-s_n/2}, T - e^{-(s+s_n)}) \\ & = & w_0(y + y_n e^{s/2}, s+s_n) \end{array} \tag{19}$$

We claim that (v_n) is a sequence of solutions of (3) which is compact in $C^3_{loc}(\mathbb{R}^N \times \mathbb{R})$. More precisely,

Lemma 2.2 *$(v_n)_{n\in\mathbb{N}}$ is a sequence of solutions of (3) with the following properties:*

i) $\lim_{n\to+\infty} |v_n(0,0)| = \kappa + \epsilon_0$ *or* $\lim_{n\to+\infty} |\nabla v_n(0,0)| = \epsilon_0$ *or* $\lim_{n\to+\infty} |\Delta v_n(0,0)| = \epsilon_0$.

ii) $\forall R > 0$, $\exists n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0$,
- $v_n(y,s)$ *is defined for* $(y,s) \in \bar{B}(0,R) \times [-R,R]$,
- $v_n \geq 0$ *and* $\|v_n\|_{L^\infty(\bar{B}(0,R)\times[-R,R])} \leq B$ *where* B *is defined in (14).*
- $\exists m(R) > 0$ *such that* $\|v_n\|_{C^3(\bar{B}(0,R)\times[-R,R])} \leq m(R)$.

Proof: *i)* v_n satisfies (3) since $w_{y_n e^{-s_n/2}}$ does the same. From (19), (17) and (18), we obtain *i)*: $\lim_{n\to+\infty} |v_n(0,0)| = \kappa + \epsilon_0$ or $\lim_{n\to+\infty} |\nabla v_n(0,0)| = \epsilon_0$ or $\lim_{n\to+\infty} |\Delta v_n(0,0)| = \epsilon_0$.

ii) Let $R > 0$.

If $\Omega = \mathbb{R}^N$, then it is obvious form (19) that v_n is defined for $(y,s) \in \bar{B}(0,R) \times [-R,R]$ for large n.

If Ω is bounded, then we can suppose that up to extracting a subsequence, $y_n e^{-s_n/2}$ converges to $y_\infty \in \bar{\Omega}$ as $n \to +\infty$. In fact $y_\infty \in \Omega$. Indeed, since $u(y_n^{(0)} e^{-s_n/2}, T - e^{-s_n}) = e^{\frac{s_n}{p-1}} v_n(0,0) \to +\infty$ as $n \to +\infty$ (or $|\nabla u(y_n^{(1)} e^{-s_n/2}, T - e^{-s_n})| = e^{s_n(\frac{1}{p-1}+\frac{1}{2})} |\nabla v_n(0,0)| \to +\infty$, or $|\Delta u(y_n^{(2)} e^{-s_n/2}, T - e^{-s_n})| = e^{s_n(\frac{1}{p-1}+1)} |\Delta v_n(0,0)| \to +\infty$), in all cases, y_∞ is a blow-up point of u. Therefore, Step 2 implies that $y_\infty \in K$ and that $B(y_\infty, \delta_0) \subset \Omega$ for some $\delta_0 > 0$. Together with (19), this implies that v_n is defined for $(y,s) \in \bar{B}(0,R) \times [-R,R]$ for large n.

From (19), (14) and the fact that $u \geq 0$, it directly follows that $v_n(y,s) \geq 0$ and $\|v_n\|_{L^\infty(\bar{B}(0,R)\times[-R,R])} \leq B$.

From lemma 2.1 and (19), it directly follows that $\forall (y,s) \in \bar{B}(0,R) \times [-R,R]$, $|v_n(y,s)| + |\nabla v_n(y,s)| + |\Delta v_n(y,s)| + |\nabla \Delta v_n(y,s)| \leq M$ and $|\frac{\partial v_n}{\partial s}| \leq M \times (1+R)$. Since $w \geq 0$, parabolic estimates and strong maximum principle imply that $\|v_n\|_{C^3(\bar{B}(o,R)\times[-R,R])} \leq m(R)$ for some $m(R) > 0$. Just take $m(R) = M \times (1 + R)$.

Now, using the compactness property of (v_n) shown in lemma 2.2, we find $v \in C^2(\mathbb{R}^N \times \mathbb{R})$ such that up to extracting a subsequence, $v_n \to v$ as $n \to +\infty$ in $C^2_{loc}(\mathbb{R}^N \times \mathbb{R})$. From lemma 2.2, it directly follows that

i) v satisfies equation (3) for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$

ii) $v \geq 0$ and $\|v\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq B$

iii) $|v(0,0)| = \kappa + \epsilon_0$ or $|\nabla v(0,0)| = \epsilon_0$ or $|\Delta v(0,0)| = \epsilon_0$ with $\epsilon_0 > 0$.

By Theorem 2, *i)* and *ii)* imply $v \equiv 0$ or $v \equiv \kappa$ or $v = \varphi(s - s_0)$ where $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$. In all cases, this contradicts *iii)*. Thus, Theorem 1 is proved. ■

Proof of Theorem 1': The proof of Theorem 1' is similar to the proof of Theorem 1. Let us sketch the main differences.

Step 1: One can remark that a uniform estimate on $E(w_{n,a}(s_0))$ where $s_0 = -\log T$ is needed. Since $\|u_0\|_{H^2(\Omega)}$ is uniformly bounded, we have the conclusion.

Step 2: One can use a uniform version of Giga and Kohn's estimates, as they are stated (for example) in [11].

Step 3: Same proof.

Proof of Corollary 1: Let us prove Corollary 1 now. We argue by contradiction and assume that for some $\epsilon_0 > 0$, there is $t_n \to T$ and $(a_n)_n$ a sequence of blow-up points of u in Ω such that

$$\forall n \in \mathbb{N},\ 0 \le u(a_n, t_n) \le (1-\epsilon_0)\kappa(T-t_n)^{-\frac{1}{p-1}}.$$

Let us give two different proofs:

Proof 1: Consider the following solution of equation (3):

$$v_n(y,s) = w_{a_n}(y, s - \log(T-t_n)).$$

From Proposition 2.1, $a_n \in K$, since it is a blow-up point of u. As before, we can use a compactness procedure on v_n to get a nonnegative bounded solution v of (3) defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $|v(0,0)| \le (1-\epsilon_0)\kappa$ and $v_n \to v$ in C^2_{loc}. Therefore, Theorem 2 implies that $v \equiv 0$ or $v = \varphi(s-s_0)$ for some $s_0 \in \mathbb{R}$. In particular, $E(v(0)) < E(\kappa)$. Since $E(v_n(0)) \to E(v(0))$ as $n \to +\infty$, we have for n large $E(w_{a_n}(-\log(T-t_n))) = E(v_n(0)) < E(\kappa)$, and in particular a_n can not be a blow-up point of u (we have from [6], for any blow-up point a of u, $E(w_a(s)) \ge E(\kappa)$ for all $s \ge -\log T$). From this fact, a contradiction follows.

Proof 2: It is a more elementary proof based on Theorem 3. Since a_n is a blow-up point and that the blow-up set is closed and bounded (see Proposition 2.1), we can assume that $a_n \to a_\infty$ where a_∞ is a blow-up point.

We know from Theorem 3 that for some $C_{\frac{\epsilon_0}{2}}$, we have $\forall x \in \Omega$, $\forall t \in [\frac{T}{2}, T)$,

$$\left|\frac{\partial u}{\partial t}(x,t) - u^p(x,t)\right| \le \frac{\epsilon_0^2}{2}|u(x,t)|^p + C_{\frac{\epsilon_0^2}{2}}. \tag{20}$$

In particular, $u(x,t) \to +\infty$ as $(x,t) \to (a_\infty, T)$ (see next subsection for a proof of Theorem 3 and this fact (22)-(23)). Let $\eta > 0$ such that

$$\forall (x,t) \in B(0,\eta) \times (T-\eta, T),\ C_{\frac{\epsilon_0^2}{2}} < \frac{\epsilon_0^2}{2}u^p(x,t). \tag{21}$$

For large n, $a_n \in B(a_\infty, \eta)$ and $t_n \in [T-\eta, T)$. Therefore (20) and (21) yield

$$\forall t \in [t_n, T),\ \frac{\partial u}{\partial t}(a_n, t) \le (1+\epsilon_0^2)u^p(a_n, t).$$

Since $0 < u(a_n, t_n) \le \kappa(1-\epsilon_0)(T-t_n)^{-\frac{1}{p-1}}$, we get by direct integration: $\forall t \in [t_n, \min(T, T^*(\epsilon_0)))$,

$$0 \le u(a_n, t) \le \kappa\left\{\frac{T-t_n}{(1-\epsilon_0)^{p-1}} - (1+\epsilon_0^2)(t-t_n)\right\}^{-\frac{1}{p-1}}$$

with $T^*(\epsilon_0) = t_n + \frac{T-t_n}{(1+\epsilon_0)(1-\epsilon_0)^{p-1}} > T$ if $\epsilon_0 < \epsilon_1(p)$ for some positive $\epsilon_1(p)$. Thus, a_n is not a blow-up point and a contradiction follows.

2.2 Global approximated behavior like an ODE

We prove Theorem 3 in this subsection. It follows from Theorem 1 and propagation of flatness (through scaling arguments) observed in [14].

Let us first show how to derive the consequences of Theorem 3 announced in the introduction:
If x_0 is a blow-up point of $u(t)$, then

$$u(x,t) \to +\infty \text{ as } (x,t) \to (x_0,T) \tag{22}$$

$$\text{and } \exists \epsilon_0 > 0 \text{ such that } \forall (x,t) \in B(x_0,\epsilon_0) \times (T-\epsilon_0,T),\ \frac{\partial u}{\partial t}(x,t) > 0. \tag{23}$$

Proof of (22) and (23):
From Theorem 3 applied with $\epsilon > 0$, there exists C_ϵ such that $\forall (x,t) \in \Omega \times [\frac{T}{2},T)$

$$\frac{\partial u}{\partial t}(x,t) \geq (1-\epsilon)u^p(x,t) - C_\epsilon. \tag{24}$$

Let A be an arbitrary large positive number satisfying

$$(1-\epsilon)A^p - C_\epsilon > 0. \tag{25}$$

From the continuity of $u(x,t)$, there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\forall x \in B(x_0,\epsilon_1)$,

$$u(x,T-\epsilon_2) > A. \tag{26}$$

From (24) and (25), we have $\forall x \in B(x_0,\epsilon_1)$, $\frac{\partial u}{\partial t}(x,T-\epsilon_2) > 0$. Now we claim that $\forall (x,t) \in B(x_0,\epsilon_1) \times (T-\epsilon_2,T)$, $u(x,t) > A$ (which yields (22) and (23) also, by (24) and (25)). Indeed, if not, then there exists $(x_1,t_1) \in B(x_0,\epsilon_1) \times (T-\epsilon_2,T)$ such that $u(x_1,t_1) \leq A$. From the continuity of u, we get $t_2 \in (T-\epsilon_2,t_1]$ such that $\forall t \in (T-\epsilon_2,t_2)$, $u(x_1,t) > A$ and $u(x_1,t_2) = A$. From (24) and (25), we have $\forall t \in (T-\epsilon_2,t_2)$, $\frac{\partial u}{\partial t}(x_1,t) > 0$, therefore, $u(x_1,t_2) > u(x_1,T-\epsilon_2) > A$ by (26). Thus, a contradiction follows, and (22) and (23) are proved.

We now prove Theorem 3.

Proof of Theorem 3: Let us argue by contradiction and suppose that for some $\epsilon_0 > 0$, there exist $(x_n,t_n)_{n\in\mathbb{N}}$ a sequence of elements of $\Omega \times [\frac{T}{2},T)$ such that $\forall n \in \mathbb{N}$,

$$|\Delta u(x_n,t_n)| \geq \epsilon_0 |u(x_n,t_n)|^p + n. \tag{27}$$

Since $\|\Delta u(t)\|_{L^\infty(\Omega)}$ is bounded on compact sets of $[\frac{T}{2},T)$, we have that $t_n \to T$ as $n \to +\infty$. We can also assume the existence of $x_\infty \in \Omega$ such that $x_n \to x_\infty$ as $n \to +\infty$. Indeed, if not, then either $d(x_n,\partial\Omega) \to 0$ (if Ω is bounded) or $|x_n| \to +\infty$ (if $\Omega = \mathbb{R}^N$) as $n \to +\infty$, and in both cases, (27) is no longer satisfied for large n, thanks to Proposition 2.1.

We claim that x_∞ is a blow-up point of u. Indeed, if not, then parabolic regularity implies the existence of a positive δ such that
$\|u(.,t)\|_{W^{2,\infty}(B(x_\infty,\delta))} \leq C$ for some positive C, which is a contradiction by (27).

Theorem 1 implies that $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}}$ is uniformly bounded, therefore, we can assume that it converges as $n \to +\infty$.
Let us consider two cases:
- Case 1: $u(x_n,t_n)(T-t_n)^{\frac{1}{p-1}} \to \kappa' > 0$ ((x_n,t_n) is in some sense in the singular region "near" (x_∞,T)). From (27), it follows that $\|\Delta u(t_n)\|_{L^\infty} \geq |\Delta u(x_n,t_n)| \geq$

$\epsilon_0 \left(\frac{\kappa'}{2}\right)^p (T-t_n)^{-\frac{p}{p-1}}$ with $t_n \to T$, which contradicts Theorem 1.

- Case 2: $u(x_n, t_n)(T-t_n)^{\frac{1}{p-1}} \to 0$ ((x_n, t_n) is in the transitory region between the singular and the regular sets).
Let us first define $(t(x_n))_n$ such that $t(x_n) \le t_n$, $t(x_n) \to T$ and

$$u(x_n, t(x_n))(T-t(x_n))^{\frac{1}{p-1}} = \kappa_0 \tag{28}$$

where $\kappa_0 \in (0,\kappa)$ satisfies $\forall t > 0$, $\forall a \in \Omega$, $\mathcal{E}_{a,t}(\kappa_0 t^{-\frac{1}{p-1}}) \le \frac{\kappa_0^2}{2(p-1)} - \frac{\kappa_0^{p+1}}{p+1} \le \frac{\sigma_0}{2}$ and σ_0 is defined in Proposition 2.2.

Step 1: Existence of $t(x_n)$

Since x_∞ is a blow-up point of u, $\lim_{t\to T} u(x_\infty, t)(T-t)^{\frac{1}{p-1}} = \kappa$. It follows that for any $\delta > 0$ small enough, there exists a ball $B(x_\infty, \delta')$ such that $\forall x \in B(x_\infty, \delta')$, $\delta^{\frac{1}{p-1}} u(x, T-\delta) \ge \frac{3\kappa+\kappa_0}{4}$. Since $x_n \to x_\infty$ as $n \to +\infty$, this implies that

$$\forall n \ge n_1,\ \delta^{\frac{1}{p-1}} u(x_n, T-\delta) \ge \frac{\kappa+\kappa_0}{2} \tag{29}$$

for some $n_1 = n_1(\delta) \in \mathbb{N}$. Since $u(x_n, t_n)(T-t_n)^{\frac{1}{p-1}} \to 0$, we have the existence of $t_\delta(x_n) \in [T-\delta, t_n] \subset [T-\delta, T)$ such that $u(x_n, t_\delta(x_n))(T-t_\delta(x_n))^{\frac{1}{p-1}} = \kappa_0$, for all $n \ge n_2(\delta)$, where $n_2(\delta) \in \mathbb{N}$. Since δ was arbitrarily small, it follows from a diagonal extraction argument that there exists a subsequence $t(x_n) \to T$ as $n \to +\infty$ such that $t(x_n) \le t_n$ and

$$u(x_n, t(x_n))(T-t(x_n))^{\frac{1}{p-1}} = \kappa_0.$$

Now, we claim that a contradiction follows if we prove the following Proposition:

Proposition 2.3 *Let*

$$v_n(\xi,\tau) = (T-t(x_n))^{\frac{1}{p-1}} u(x_n + \xi\sqrt{T-t(x_n)}, t(x_n) + \tau(T-t(x_n))). \tag{30}$$

Then, v_n is a solution of (1) for $\tau \in [0,1)$, and there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$,

$$\forall \tau \in [0,1),\ |\Delta v_n(0,\tau)| \le \frac{\epsilon_0}{2} |v_n(0,\tau)|^p. \tag{31}$$

Indeed, from (31) and (30), we obtain: $\forall n \ge n_0$, $\forall t \in [t(x_n), T)$,
$|\Delta u(x_n, t)| = (T-t(x_n))^{-(\frac{1}{p-1}+1)} |\Delta_\xi v_n(0, \tau(t,n))|$
$\le \frac{\epsilon_0}{2}(T-t(x_n))^{-\frac{p}{p-1}} |v_n(0,\tau(t,n))|^p = \frac{\epsilon_0}{2}|u(x_n,t)|^p$ with $\tau(t,n) = \frac{t-t(x_n)}{T-t(x_n)}$, which contradicts (27), since $t_n \ge t(x_n)$. Thus, Theorem 3 is proved.

Step 2: Flatness of v_n

In this Step we prove Proposition 2.3.

We claim that the following lemma concludes the proof of Proposition 2.3:

Lemma 2.3 *i) $\forall \delta_0 > 0$, $\forall A > 0$, $\exists n_3(\delta_0, A) \in \mathbb{N}$ such that $\forall n \ge n_3(\delta_0, A)$, for all $|\xi| \le A$ and $\tau \in [0, \frac{3}{4}]$, $|v_n(\xi, 0) - \kappa_0| \le \delta_0$, $|\nabla_\xi v_n(\xi,\tau)| \le \delta_0$ and $|\Delta_\xi v_n(\xi,\tau)| \le \delta_0$.*

ii) $\forall \epsilon > 0$, $\forall A > 0$, $\exists n_4(\epsilon, A) \in \mathbb{N}$ such that $\forall n \ge n_4$, $\forall \tau \in [0,1)$, for $|\xi| \le \frac{A}{4}$, $|v_n(\xi,\tau) - \hat{v}(\tau)| \le \epsilon$, $|\nabla v_n(\xi,\tau)| \le \epsilon$ and $|\Delta v_n(\xi,\tau)| \le \epsilon$ where $\hat{v}(\tau) = \kappa\left(\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - \tau\right)^{-\frac{1}{p-1}}$ is a solution of $\frac{d\hat{v}}{d\tau} = \hat{v}^p$ with $\hat{v}(0) = \kappa_0$.

Indeed, if ϵ is small enough and n is large enough, then $\forall \tau \in [0,1)$, $v_n(0,\tau) \geq \frac{1}{2}\hat{v}(0) = \frac{\kappa_0}{2}$ and $|\Delta v_n(0,\tau)| \leq \left(\frac{\kappa_0}{2}\right)^p \frac{\epsilon_0}{2} \leq \frac{\epsilon_0}{2}|v_n(0,\tau)|^p$.

Proof of lemma 2.3: $i)$ Let $\delta_0 > 0$ and $A > 0$. From (28) and (30), we have: for all $|\xi| \leq A$ and $\tau \in [0, \frac{3}{4}]$:
$v_n(0,0) = \kappa_0$,
$|v_n(\xi,0) - v_n(0,0)| \leq (T - t(x_n))^{\frac{1}{p-1}+\frac{1}{2}} A \|\nabla u(t(x_n))\|_{L^\infty(\Omega)}$,
$\nabla v_n(\xi,\tau) = (T - t(x_n))^{\frac{1}{p-1}+\frac{1}{2}} \nabla u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$
$= \left(\frac{1}{1-\tau}\right)^{\frac{1}{p-1}+\frac{1}{2}} (T - (t(x_n) + \tau\,(T - t(x_n))))^{\frac{1}{p-1}+\frac{1}{2}} \times$
$\nabla u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$ and
$\Delta v_n(\xi,\tau) = (T - t(x_n))^{\frac{1}{p-1}+1} \Delta u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right) =$
$\left(\frac{1}{1-\tau}\right)^{\frac{1}{p-1}+1} (T - (t(x_n) + \tau\,(T - t(x_n))))^{\frac{1}{p-1}+1} \times$
$\Delta u\left(x_n + \xi\sqrt{T - t(x_n)}, t(x_n) + \tau(T - t(x_n))\right)$.

Since $\tau \leq \frac{3}{4}$, $t(x_n) \to T$ as $n \to +\infty$, and $(T-t)^{\frac{1}{p-1}+\frac{1}{2}}\|\nabla u(t)\|_{L^\infty(\Omega)} + (T-t)^{\frac{1}{p-1}+1}\|\Delta u(t)\|_{L^\infty(\Omega)} \to 0$ as $t \to T$ (Theorem 1), $i)$ is proved.

$ii)$ From $i)$ and continuity arguments, it follows that for all $|\xi| \leq A$, $\mathcal{E}_{\xi,1}(v_n(0)) \leq 2\mathcal{E}_{\xi,1}(\kappa_0) \leq \sigma_0$ for n large enough, by definition of κ_0. Therefore, from Proposition 2.2 (applied with $\delta = 1$ and using translation invariance), we have $\forall \tau \in [\frac{1}{2}, 1)$, $\forall |\xi| \leq \frac{A}{2}$, $|v_n(\xi,\tau)| \leq M(p)$.

By classical parabolic arguments, we get

$$\forall \tau \in [\frac{3}{4}, 1),\ \forall |\xi| \leq \frac{A}{3},\ |v_n| + |\nabla v_n| + |\Delta v_n| \leq M(p). \tag{32}$$

Now, using $i)$, (32) and classical estimates for the heat flow, we get for all $\epsilon > 0$: $\forall |\xi| \leq \frac{A}{4}$, $\forall \tau \in [0,1)$, $|\nabla v_n(\xi,\tau)| \leq \epsilon$ and $|\Delta v_n(\xi,\tau)| \leq \epsilon$ if $n \geq n_5(\epsilon, A)$.

Since v_n is a solution of equation (1), combining this with $i)$ and ODE estimates yields for all $\epsilon > 0$: $\forall |\xi| \leq \frac{A}{4}$, $\forall \tau \in [0,1)$, $|v_n(\xi,\tau) - \hat{v}(\tau)| \leq \epsilon$ if $n \geq n_6(\epsilon, A)$. This concludes the proof of $ii)$. ■

3 Classification of connections between critical points of equation (3) in L^∞_{loc}

We prove Theorem 2 and Corollaries 2 and 3 in this section.

We first prove Theorem 2, and then we show how Corollaries 2 and 3 can be deduced from Theorem 2.

Proof of Theorem 2: We assume that $1 < p$ and $(N-2)p < N+2$, and consider $w(y,s)$ a nonnegative global bounded solution of (3) defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. Our goal is to show that w depends only on time s.

We proceed in 5 steps.

In Step 1, we show that w has a limit $w_{\pm\infty}$ as $s \to \pm\infty$, where $w_{\pm\infty}$ is a critical point of (3), that is $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \kappa$. We focus then on the non trivial case, that is $w_{-\infty} \equiv \kappa$ and $w_{+\infty} \equiv 0$.

In Step 2, we investigate the linear problem around κ, as $s \to -\infty$, and show that w would behave at most in three ways.

In Step 3, we show that among these three ways we have the situation $w(y,s) = \varphi(s - s_0)$ with $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$. We then show (respectively in Step 4 and in Step 5) that the two other ways actually can not occur, we find in fact a contradiction through a blow-up argument for $w(s)$ using the geometrical transformation:

$$a \to w_a \text{ defined by } w_a(y,s) = w(y + ae^{\frac{s}{2}}, s) \tag{33}$$

(w_a is also a solution of (3)) and a blow-up criterion for equation (3).

Step 1: Behavior of w as $s \to \pm\infty$

This step can be found in Giga and Kohn [6]. The results are mainly consequences of parabolic estimates and the gradient structure of equation (3). Let us recall them briefly. We first restate lemma 2.1 of section 2:

Lemma 3.1 (Parabolic estimates) *There is a positive constant M such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$,*

$$|w(y,s)| + |\nabla w(y,s)| + |\Delta w(y,s)| \le M \text{ and } \left|\frac{\partial w}{\partial s}(y,s)\right| \le M(1+|y|).$$

Lemma 3.2 (Stationary solutions) *Assume $p \le (N+2)/(N-2)$ or $N \le 2$. Then the only nonnegative bounded global solutions in $\mathbb{R}^N$ of*

$$0 = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w \tag{34}$$

are the trivial ones: $w \equiv 0$ and $w \equiv \kappa$.

Proof: The following Pohozaev identity can be derived for each bounded solutions of equation (3) in $\mathbb{R}^N$ (see Proposition 2 in [6]):

$$(N+2-p(N-2)) \int |\nabla w|^2 \rho dy + \frac{p-1}{2} \int |y|^2 |\nabla w|^2 \rho dy = 0.$$

Hence, for $(N-2)p \le N+2$, w is constant. Thus, $w \equiv 0$ or $w \equiv \kappa$. ■

Lemma 3.3 (Gradient structure) *Assume $p < (N+2)/(N-2)$ or $N \le 2$. We define for each w solution of (3)*

$$E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 \rho dy + \frac{1}{2(p-1)} \int_{\mathbb{R}^N} |w|^2 \rho dy - \frac{1}{p+1} \int_{\mathbb{R}^N} |w|^{p+1} \rho dy \tag{35}$$

$$\text{where } \rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{N/2}}. \tag{36}$$

Then, $\forall s_1, s_2 \in \mathbb{R}$,

$$\int_{s_1}^{s_2} \int_{\mathbb{R}^N} \left|\frac{\partial w}{\partial s}\right|^2 \rho dy ds = E(w(s_1)) - E(w(s_2)) \tag{37}$$

Outline of the proof: (*see* Proposition 3 in [6] for more details).

One may multiply equation (3) by $\frac{\partial w}{\partial s}\rho$ and integrate over the ball $B(0,R)$ with $R>0$. Then, using lemma 3.1 and the dominated convergence theorem yields the result. ■

Proposition 3.1 (Limit of w as $s\to\pm\infty$) *Assume $p<(N+2)/(N-2)$ or $N\le 2$. Let w be a bounded nonnegative global solution of (3) in $\mathbb{R}^{N+1}$. Then $w_{+\infty}(y)=\lim\limits_{s\to+\infty} w(y,s)$ exists and equals 0 or κ. The convergence is uniform on every compact subset of $\mathbb{R}^N$. The corresponding statements hold also for the limit $w_{-\infty}(y)=\lim\limits_{s\to-\infty} w(y,s)$.*

Outline of the proof: (*see* Propositions 4 and 5 in [6] for more details).

Let (s_j) be a sequence tending to $+\infty$, and let $w_j(y,s)=w(y,s+s_j)$. From lemma 3.1, (w_j) converges uniformly on compact sets to some $w_{+\infty}(y,s)$ and $\nabla w_j\to\nabla w_{+\infty}$ a.e. Assuming that $s_{j+1}-s_j\to+\infty$, one can use lemma 3.3 to show that w_j does not depend on s. Therefore, $w_{+\infty}\equiv 0$ or $w_{+\infty}=\kappa$ by lemma 3.2. The continuity of w then asserts that $w_{+\infty}$ does not depend on the choice of the subsequence (s_j). The analysis in $-\infty$ is completely parallel. ■

According to (37) (with $s_1\to-\infty$ and $s_2\to+\infty$), there are only two cases:
- $E(w_{-\infty})-E(w_{+\infty})=0$: hence, $\frac{\partial w}{\partial s}\equiv 0$. Therefore, w is a bounded global solution of (34). Thus, $w\equiv 0$ or $w\equiv\kappa$ according to lemma 3.2. This case has been treated by Giga and Kohn in [6].
- $E(w_{-\infty})-E(w_{+\infty})>0$: since $E(\kappa)=(\frac{1}{2}-\frac{1}{p+1})\kappa^{p+1}\int\rho dy>0=E(0)$, we have $(w_{-\infty},w_{+\infty})=(\kappa,0)$. It remains to treat this case in order to finish the proof of Theorem 2.

In the following steps, we consider the case

$$(w_{-\infty},w_{+\infty})=(\kappa,0).$$

Step 2: Classification of the behavior of w as $s\to-\infty$:

Since w is globally bounded in L^∞ and $w\to\kappa$ as $s\to-\infty$, uniformly on compact subsets of $\mathbb{R}^N$, we have $\lim\limits_{s\to-\infty}\|w-\kappa\|_{L^2_\rho}=0$ where L^2_ρ is the L^2-space associated to the Gaussian measure $\rho(y)dy$ and ρ is defined in (36).

In this part, we classify the L^2_ρ behavior of $w-\kappa$ as $s\to-\infty$. Let us introduce $v=w-\kappa$. From (3), v satisfies the following equation: $\forall(y,s)\in\mathbb{R}^{N+1}$,

$$\frac{\partial v}{\partial s}=\mathcal{L}v+f(v) \tag{38}$$

where

$$\mathcal{L}v=\Delta v-\frac{1}{2}y.\nabla v+v \text{ and } f(v)=|v+\kappa|^{p-1}(v+\kappa)-\kappa^p-p\kappa^{p-1}v. \tag{39}$$

Since w is bounded in L^∞, we can assume $|v(y,s)|\le M$, and then $|f(v)|\le Cv^2$ with $C=C(M)$.

$\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L})\subset L^2_\rho$. Its spectrum is

$$spec(\mathcal{L})=\{1-\frac{m}{2}|m\in\mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

- $N = 1$:
  All the eigenvalues of $\mathcal{L}$ are simple. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{40}$$

  h_m satisfies $\int h_n h_m \rho dy = 2^n n! \delta_{nm}$. Let us introduce

$$k_m = h_m / \|h_m\|^2_{L^2_\rho}. \tag{41}$$

- $N \geq 2$:
  We write the spectrum of $\mathcal{L}$ as

$$spec(\mathcal{L}) = \{1 - \frac{m_1 + ... + m_N}{2} | m_1, ..., m_N \in \mathbb{N}\}.$$

  For $(m_1, ..., m_N) \in \mathbb{N}^N$, the eigenfunction corresponding to $1 - \frac{m_1 + ... + m_N}{2}$ is

$$y \longrightarrow h_{m_1}(y_1)...h_{m_N}(y_N),$$

  where h_m is defined in (40). In particular,

  *1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is

$$H_0(y) = 1, \tag{42}$$

  *$\frac{1}{2}$ is of multiplicity N, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y) | i = 1, ..., N\}$, with $H_{1,i}(y) = h_1(y_i)$; we note

$$H_1(y) = (H_{1,1}(y), ..., H_{1,N}(y)), \tag{43}$$

  *0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y) | i, j = 1, ..., N, i \leq j\}$, with $H_{2,ii}(y) = h_2(y_i)$, and for $i < j$, $H_{2,ij}(y) = h_1(y_i) h_1(y_j)$; we note

$$H_2(y) = (H_{2,ij}(y), i \leq j). \tag{44}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of L^2_ρ, we expand v as follows:

$$v(y,s) = \sum_{m=0}^{2} v_m(s).H_m(y) + v_-(y,s) \tag{45}$$

where
$v_0(s)$ is the projection of v on H_0,
$v_{1,i}(s)$ is the projection of v on $H_{1,i}$, $v_1(s) = (v_{1,i}(s), ..., v_{1,N}(s))$, $H_1(y)$ is given by (43),
$v_{2,ij}(s)$ is the projection of v on $H_{2,ij}$, $i \leq j$, $v_2(s) = (v_{2,ij}(s), i \leq j)$, $H_2(y)$ is

given by (44),
$v_-(y,s) = P_-(v)$ and P_- the projector on the negative subspace of $\mathcal{L}$.

With respect to the positive, null and negative subspaces of $\mathcal{L}$, we write

$$v(y,s) = v_+(y,s) + v_{null}(y,s) + v_-(y,s) \tag{46}$$

where $v_+(y,s) = P_+(v) = \sum_{m=0}^{1} v_m(s).H_m(y)$,
$v_{null}(y,s) = P_{null}(v) = v_2(s).H_2(y)$, P_+ and P_{null} are the L^2_ρ projectors respectively on the positive subspace and the null subspace of $\mathcal{L}$.

Now, we show that as $s \to -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (45) of v in L^2_ρ. At this level, we are not able to use a center manifold theory to get the result (see [3] page 834-835 for more details). In some sense, we are not able to say that the nonlinear terms in the function of space are small enough. However, using similar techniques as in [3], we are able to prove the result. We have the following:

Proposition 3.2 (Classification of the behavior of $v(y,s)$ as $s \to -\infty$) *As $s \to -\infty$, one of the following situations occurs:*
i) $|v_1(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(v_0(s))$,
ii) $|v_0(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(|v_1(s)|)$,
iii) $\|v_+(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(\|v_{null}(y,s)\|_{L^2_\rho})$.

Proof: See Appendix A.

Now we handle successively the three cases suggested by proposition 3.2 to show that only case $i)$ occurs.

In case $i)$, we end up to show that $w(y,s) = \varphi(s-s_0)$ for some $s_0 \in \mathbb{R}$, where φ is defined in (9). In cases $ii)$ and $iii)$, we show that the solutions satisfy through an elementary geometrical transformation a blow-up condition for equation (3) considered for increasing s, which contradicts their boundedness, and concludes the proof of Theorem 2.

Step 3: Case $i)$ of Proposition 3.2: $\exists s_0 \in \mathbb{R}$ such that $w(y,s) = \varphi(s-s_0)$

Proposition 3.3 *Suppose that $|v_1(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho}$ $= o(v_0(s))$ as $s \to -\infty$, then there exists $s_0 \in \mathbb{R}$ such that:*
i) $\forall \epsilon > 0$, $v_0(s) = -\frac{\kappa}{p-1}e^{s-s_0} + O(e^{(2-\epsilon)s})$ *as* $s \to -\infty$,
ii) $\forall (y,s) \in \mathbb{R}^{N+1}$ $w(y,s) = \varphi(s-s_0)$ *where* $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$.

Remark: This proposition asserts that if a solution of (38) behaves like a constant independent of y (that is like $v_0(s)$), then it is exactly a constant.

Proof: $i)$ See Step 3 of Appendix A and take $s_0 = -\log(-\frac{(p-1)C_0}{\kappa})$.

We remark that we already know a solution of equation (38) which behaves like $i)$. Indeed, $\varphi(s-s_0) - \kappa = (\varphi(s-s_0) - \kappa)h_0$ is a solution of (38) which satisfies

$$\varphi(s-s_0) - \kappa = -\frac{\kappa}{p-1}e^{s-s_0} + O(e^{(2-\epsilon)s}) \text{ as } s \to -\infty.$$

From a dimension argument, we expect that for each parameter, there is at most one solution such that:

$$v_0(s) \sim -\frac{\kappa}{p-1}e^{s-s_0} \text{ as } s \to -\infty.$$

(if for example, center manifold analysis applies). We propose to prove this fact.

In other words, our goal is to show that

$$\forall (y,s) \in \mathbb{R}^{N+1}, v(y,s) = \varphi(s-s_0) - \kappa.$$

Since (38) is invariant under translations in time, we can assume $s_0 = 0$ without loss of generality.

For this purpose, we introduce

$$V(y,s) = v(y,s) - (\varphi(s) - \kappa) = w(y,s) - \varphi(s). \tag{47}$$

From (3), V satisfies the following equation:

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V)$$

where $\mathcal{L} = \Delta - \frac{1}{2} y.\nabla + 1$, $l(s) = -\frac{pe^s}{(p-1)(1+e^s)}$ and $F(V) = |\varphi + V|^{p-1}(\varphi + V) - \varphi^p - p\varphi^{p-1}V$. Note that $\forall s \le 0$, $|F(V)| \le C|V|^2$ where $C = C(M)$ and $M \ge \|v\|_{L^\infty}$.

We know from Step 3 in Appendix A that

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}),\ \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty.$$

The following Proposition asserts that $V \equiv 0$, which concludes the proof of Proposition 3.3:

Proposition 3.4 *Let V be an L^∞ solution of*

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V)$$

defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $V \to 0$ as $s \to \pm\infty$ uniformly on compact sets of $\mathbb{R}^N$,

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty.$$

Then $V \equiv 0$.

Proof: see Appendix B.

Step 4: Irrelevance of the case where $v_1(s)$ is preponderant

In this case *ii*) of Proposition 3.3, we use the main term in the expansion of $v(s)$ as $s \to -\infty$ to find a_0 and s_0 such that

$$\int w_{a_0}(y,s_0)\rho(y)dy > \kappa \tag{48}$$

where w_{a_0} is defined in (33). Since $w \ge 0$, we find that (48) implies that w_{a_0} (which is also a solution of (3)) blows-up in finite time $S > s_0$ (and so does w), which contradicts the fact that w is globally bounded. It is in fact mainly the only place where the hypothesis

$$w \ge 0$$

is used. More precisely, let us state the following Proposition:

Proposition 3.5 (A blow-up criterion for equation (3)) *Consider $W \ge 0$ a solution of (3) and suppose that for some $s_0 \in \mathbb{R}$, $\int W(y,s_0)\rho(y)dy > \int \kappa\rho dy = \kappa$. Then W blows-up in finite time $S > s_0$.*

Proof: We argue by contradiction and suppose that W is defined for all $s \in [s_0, +\infty)$. If $V = W - \kappa$, then V satisfies equation (38). Let us define

$$z_0(s) = \int V(y,s)\rho(y)dy.$$

Integrating (38) with respect to ρdy, we obtain

$$z_0'(s) = z_0(s) + \int f(V(y,s))\rho dy$$

$$\text{where} \qquad f(x) = (\kappa + x)^p - \kappa^p - p\kappa^{p-1}x \text{ for } \kappa + x \geq 0.$$

It is obvious that f is nonnegative and convex on $[-\kappa, +\infty)$. Since $W = \kappa + V \geq 0$, $\rho \geq 0$ and $\int \rho dy = 1$, we have the following Jensen's inequality:

$$\int f(V(y,s))\rho dy \geq f(\int V(y,s)\rho dy) = f(z_0(s)).$$

Therefore,

$$z_0'(s) \geq z_0(s) + f(z_0(s)). \tag{49}$$

Since $f(x) > 0$ for $x > 0$ (f is strictly convex and $f(0) = f'(0) = 0$) and $z_0(s_0) > 0$ by the hypothesis, by classical arguments, we have $\forall s \geq s_0$, $z_0'(s) \geq 0$, therefore, $\forall s \geq s_0$, $z_0(s) > 0$. By direct integration, we have $\forall s \geq s_0$,

$$s - s_0 \leq \int_{z_0(s_0)}^{z_0(s)} \frac{dx}{f(x)} \leq \int_{z_0(s_0)}^{+\infty} \frac{dx}{f(x)}.$$

Since $\frac{1}{f(x)} \sim \frac{1}{|x|^p}$ as $s \to +\infty$, a contradiction follows and Proposition 3.5 is proved. ∎

Proposition 3.6 (Case where $v_1(s)$ is preponderant) *Suppose that* $|v_0(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(|v_1(s)|)$, *then:*

i) $\exists C_1 \in \mathbb{R}^N \backslash \{0\}$ *such that* $v_0(s) \sim \frac{p}{\kappa}|C_1|^2 s e^s$ *and* $v_1(s) \sim C_1 e^{s/2}$ *as* $s \to -\infty$.

ii) $\exists a_0 \in \mathbb{R}^N$, $\exists s_0 \in \mathbb{R}$ *such that* $\int w_{a_0}(y,s_0)\rho(y)dy > \kappa$ *where* w_{a_0} *introduced in (33) is a solution of equation (3) defined for* $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ *satisfying* $\|w_{a_0}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})} \leq B$.

From Proposition 3.5, *ii)* is a contradiction.

Remark: w_a has a geometrical interpretation in terms of $w(y,s)$. Indeed, from $w(y,s)$, we introduce $u(x,t)$ (as in (2)) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, 0)$ by:

$$x = \frac{y}{\sqrt{-t}}, \; s = -\log(-t), \; u(x,t) = (-t)^{-\frac{1}{p-1}} w(y,s).$$

Now, if we define $\hat{w}_a(y,s)$ from $u(x,t)$ by (2) as

$$x = \frac{y-a}{\sqrt{-t}}, \; s = -\log(-t), \; \hat{w}_a(y,s) = (-t)^{\frac{1}{p-1}} u(x,t),$$

then, $\hat{w}_a \equiv w_a$.

Proof of Proposition 3.6: *i*) follows from Step 3 in Appendix A.

Therefore, we prove *ii*). It is easy to check that w_a satisfies (3). Moreover, from (33) we get $\|w_a\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} = \|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq B$. We want to show that there exist $a \in \mathbb{R}^N$ and $s_0 \in \mathbb{R}$ such that $\int w_a(y,s_0)\rho(y)dy > \kappa$. From (33), we have:

$\int w_a(y,s)\rho dy = \int w(y+ae^{s/2},s)\rho dy$.

Let us note $\alpha = ae^{s/2}$. The conclusion follows if we show that there exist $s_0 \in \mathbb{R}$ and $\alpha(s_0) \in \mathbb{R}^N$ such that $\int w(y+\alpha(s_0),s_0)\rho dy > \kappa$.

For this purpose, we search an expansion for $\int w(y+\alpha,s)\rho dy$ as $s \to -\infty$ and $\alpha \to 0$.

$\int w(y+\alpha,s)\rho dy = \int w(y,s)\rho(y-\alpha)dy = \kappa + \int v(y,s)\frac{e^{-\frac{|y-\alpha|^2}{4}}}{(4\pi)^{N/2}}dy$

$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)e^{\frac{\alpha.y}{2}}dy$

$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)\left(1+\frac{\alpha.y}{2}+\frac{(\alpha.y)^2}{8}\int_0^1(1-\xi)e^{\xi\frac{\alpha.y}{2}}d\xi\right)dy$

$= \kappa + (1+O(|\alpha|^2))\left(v_0(s)+\alpha.v_1(s)+(I)\right)$

where $(I) = e^{-\frac{|\alpha|^2}{4}}\int dy v(y,s)\rho(y)\frac{(\alpha.y)^2}{8}\int_0^1 d\xi(1-\xi)e^{\xi\frac{\alpha.y}{2}}$.

Using Schwartz's inequality, we have

$|(I)| \leq \left(\int v(y,s)\rho(y)dy\right)^{1/2}\left(\int dy\frac{(\alpha.y)^4}{64}\rho(y)\left(\int_0^1 d\xi(1-\xi)e^{\xi\frac{\alpha.y}{2}}\right)^2\right)^{1/2}$

$\leq \|v(s)\|_{L^2_\rho}\times\frac{|\alpha|^2}{8}\left(\int dy|y|^4\rho(y)\left(\int_0^1 d\xi(1-\xi)e^{\frac{|y|}{2}}\right)^2\right)^{1/2}$

$\leq \|v(s)\|_{L^2_\rho}\times\frac{|\alpha|^2}{16}\left(\int dy|y|^4\rho(y)e^{|y|}\right)^{1/2} = C|\alpha|^2\|v(s)\|_{L^2_\rho}$.

Therefore, using the fact that $\|v(s)\|_{L^2_\rho} \sim 2|v_1(s)| = O(e^{s/2})$ and *i*), we get:

$\int w(y+\alpha,s)\rho dy = \kappa + v_0(s) + \alpha.v_1(s) + O(|\alpha|^2e^{s/2})$

$= \kappa + \frac{p}{\kappa}|C_1|^2se^s + o(se^s) + \alpha.C_1e^{s/2} + o(|\alpha|e^{s/2})$.

Now, if we make $\alpha = \alpha(s) = -\frac{1}{s}\frac{C_1}{|C_1|}$ and take $-s$ large enough, then $\int w(y+\alpha(s),s) - \kappa \geq \frac{1}{2}\alpha(s).C_1e^{s/2} = -\frac{e^{s/2}}{2s}|C_1| > 0$, and the existence of a_0 and s_0 is proved.

This concludes the proof of Proposition 3.6. ■

Step 5: Irrelevance of the case where $v_2(s)$ is preponderant

As in the previous part, we use the information given by the linear theory at $-\infty$ to find a contradiction in the case where *iii*) holds in Proposition 3.2.

Proposition 3.7 (Case where $v_2(s)$ is preponderant) *Assume that* $\|v_+(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(\|v_{null}(y,s)\|_{L^2_\rho})$, *then:*

i) there exists $\delta \geq 0$, $k \in \{0,1...,N-1\}$ *and* Q *an orthonormal* $N\times N$ *matrix such that*

$$\begin{aligned} v_{null}(y,s) &= y^TA(s)y - 2trA(s) \\ \textit{where } A(s) &= -\frac{\kappa}{4ps}A_0 + O\Big(\frac{1}{s^{1+\delta}}\Big) \textit{ as } s\to-\infty, \end{aligned}$$

$$A_0 = Q\begin{pmatrix} I_{N-k} & 0 \\ 0 & 0\end{pmatrix}Q^{-1}$$

and I_{N-k} *is the* $(N-k)\times(N-k)$ *identity matrix. Moreover,*

$$\|v(s)\|_{L^2_\rho} = -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right),\ v_0(s) = O\Big(\frac{1}{s^2}\Big) \textit{ and } v_1(s) = O\Big(\frac{1}{s^2}\Big).$$

ii) $\exists a_0 \in \mathbb{R}^N$, $\exists s_0 \in \mathbb{R}$ *such that* $\int w_{a_0}(y,s_0)\rho(y)dy > \kappa$ *where* w_{a_0} *defined in (33) is a solution of equation (3) satisfying* $\|w_{a_0}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \leq B$.

From *ii*) and Proposition 3.5, a contradiction follows.

Proof of i) of Proposition 3.7:

The first part of the proof follows as before the ideas of Filippas and Kohn in [3]. Then, we carry on the proof similarly as Filippas and Liu did in [4] for the same equation when the null mode dominates as $s \to +\infty$. Since the used techniques are the same than in [3] and [4], we leave the proof in Appendix C.

Proof of ii) of Proposition 3.7:

We proceed exactly in the same way as for the proof of *ii*) of Proposition 3.6. w_a satisfies equation (3), and the L^∞ bound on w_a follows as before.

By setting $\alpha = ae^{s/2}$, the proof reduces then to find s_0 and $\alpha = \alpha(s_0)$ such that $\int w(y+\alpha(s_0),s_0)\rho dy > \kappa$.

For this purpose, we search an expansion for $\int w(y+\alpha,s)\rho dy$ as $s \to -\infty$ and $\alpha \to 0$.

$\int w(y+\alpha,s)\rho dy = \int w(y,s)\rho(y-\alpha)dy = \kappa + \int v(y,s)\frac{e^{-\frac{|y-\alpha|^2}{4}}}{(4\pi)^{N/2}}dy$
$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)e^{\frac{\alpha.y}{2}}dy$
$= \kappa + e^{-\frac{|\alpha|^2}{4}}\int v(y,s)\rho(y)\left(1+\frac{\alpha.y}{2}+\frac{(\alpha.y)^2}{8}+\frac{(\alpha.y)^3}{16}\int_0^1(1-\xi)^2 e^{\xi\frac{\alpha.y}{2}}d\xi\right)dy.$

We write

$$\int w(y+\alpha,s)\rho dy = \kappa + (I) + (II), \tag{50}$$

where

$(I) = e^{-\frac{|\alpha|^2}{4}}(v_0(s)+\alpha.v_1(s)) + e^{-\frac{|\alpha|^2}{4}}\int dy v(y,s)\rho(y)\frac{(\alpha.y)^3}{16}\int_0^1 d\xi(1-\xi)^2 e^{\xi\frac{\alpha.y}{2}}$

and $(II) = \frac{1}{8}e^{-\frac{|\alpha|^2}{4}}\int v(y,s)(\alpha.y)^2\rho(y)dy$.

From *i*) of Proposition 3.7 and Schwartz's inequality, we have

$$|(I)| \leq \frac{C}{s^2} + C\frac{|\alpha|^3}{|s|}. \tag{51}$$

Since $v = v_- + v_{null} + v_+ = v_- + v_{null} + v_1.y + v_0$, we have from the orthogonality of v_- and $v_{null}+v_+$:

$(II) = \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\int v(y,s)(\alpha.y)^2\rho dy$
$= \frac{e^{-\frac{|\alpha|^2}{4}}}{8}(v_0(s)\int(\alpha.y)^2\rho dy + v_1(s).\int y(\alpha.y)^2\rho dy) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\int v_{null}(\alpha.y)^2\rho dy$
$= v_0(s)O(|\alpha|^2) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\int(y^TA(s)y - 2trA(s))(\alpha.y)^2\rho dy$
$= O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\frac{\kappa}{4p|s|}\int(y^TA_0y - 2trA_0)(\alpha.y)^2\rho dy$

for some $\delta > 0$, according to *i*) of Proposition 3.7, with

$$A_0 = Q\begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}.$$

With the change of variable $y = Q^{-1}z$ (Q is an orthonormal matrix) we write:

$(II) = O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + \frac{e^{-\frac{|\alpha|^2}{4}}}{8}\frac{\kappa}{4p|s|}\int\sum_{i=1}^{N-k}(z_i^2-2)\times(Q\alpha.z)^2\rho(z)dz,$

therefore,

$$(52)\quad (II) = \frac{\kappa}{4p|s|}\sum_{i=1}^{N-k}\int (z_i^2-2)(Q\alpha.z)^2\rho dz + O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + O\left(\frac{|\alpha|^4}{|s|}\right).$$

Gathering (50), (51) and (52), we write:
$\int w(y+\alpha,s)\rho dy$

$$= \kappa + \frac{\kappa}{4p|s|}\sum_{i=1}^{N-k}\int (z_i^2-2)(Q\alpha.z)^2\rho dz + O(\frac{1}{s^2}) + O\left(\frac{|\alpha|^2}{|s|^{1+\delta}}\right) + O\left(\frac{|\alpha|^3}{|s|}\right).$$

Now, if we take $\alpha = \alpha(s) = \frac{1}{|s|^{1/4}}Q^{-1}e^1$ where $e^1 = (1,0,...,0)$, then

$$\int w(y+\alpha(s),s)\rho dy = \kappa + \frac{\kappa}{4p|s|^{3/2}}\times 8 + O\left(\frac{1}{|s|^{3/2+\delta}}\right).$$

If we take $-s$ large enough, and $a(s) = e^{-s/2}\alpha(s)$, then

$$\int w(y+\alpha(s)e^{s/2},s) > \kappa.$$

This concludes the proof of $ii)$ of Proposition 3.7 and the proof of Theorem 2.

We now prove Corollaries 1 and 2:

Proof of Corollary 2:

We consider w a nonnegative solution of (3) defined for $(y,s)\in\mathbb{R}^N\times(-\infty,s^*)$ where $s^*\in\mathbb{R}\cup\{+\infty\}$. We assume that there is a constant C_0 such that

$$(53)\qquad \forall a\in\mathbb{R}^N,\ \forall s\le s^*,\ E_a(w(s))\le C_0$$

where E_a is defined in (10).

Through some geometrical transformations, we define below $\hat{w}$, a solution of (3) defined on $\mathbb{R}^N\times\mathbb{R}$, which satisfies the hypotheses of Theorem 2. Then, we deduce the characterization of w from the one given in Theorem 2 for $\hat{w}$.

Let us define $u(t)$ a solution of (1) by:

$$(54)\qquad y=\frac{x}{\sqrt{-t}},\ s=-\log(-t),\ u(x,t)=(-t)^{-\frac{1}{p-1}}w(y,s)$$

where $(x,t)\in\mathbb{R}^N\times(-\infty,T^*)$ with $T^*=-e^{-s^*}$ if s^* is finite and $T^*=0$ if $s^*=+\infty$. Then we introduce $\hat{w}$ a solution of (3):

$$(55)\qquad y=\frac{x}{\sqrt{T^*-t}},\ s=-\log(T^*-t),\ \hat{w}(y,s)=(T^*-t)^{\frac{1}{p-1}}u(x,t)$$

defined for $(y,s)\in\mathbb{R}^N\times\mathbb{R}$. We have then $\forall(y,s)\in\mathbb{R}^N\times(-\infty,s^*)$,

$$(56)\qquad w(y,s)=(1+T^*e^s)^{-\frac{1}{p-1}}\hat{w}(\frac{y}{\sqrt{1+T^*e^s}},s-\log(1+T^*e^s)).$$

We claim that $\hat{w}\in L^\infty(\mathbb{R}^N\times\mathbb{R})$. Indeed, from (53), (54) and $i)$ of Proposition 2.2, we have $\forall(x,t)\in\mathbb{R}^N\times(-\infty,T^*)$, $|u(x,t)|\le M(C_0)(T^*-t)^{-\frac{1}{p-1}}$. Hence, (55) implies that $\forall(y,s)\in\mathbb{R}^N\times\mathbb{R}$, $|w(y,s)|\le M(C_0)$.

Since w is nonnegative, $\hat{w}$ is also nonnegative, and then, by Theorem 2 we have:
either $\hat{w} \equiv 0$, or $\hat{w} \equiv \kappa$
or $\hat{w}(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$, where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$.
Therefore, by (56), we have:
either $w \equiv 0$, or $w(y,s) = \kappa(1 - e^{s-s^*})^{-\frac{1}{p-1}}$
or $w(y,s) = (1-e^{s-s^*})^{-\frac{1}{p-1}} \kappa \left(1 + \exp(s - \log(1 - e^{s-s^*} - s_0))\right)^{-\frac{1}{p-1}}$
$= \kappa \left(1 + e^s(e^{-s_0} - e^{-s^*})\right)^{-\frac{1}{p-1}}$.
Since s_0 is arbitrary in $\mathbb{R}$, this concludes the proof of Corollary 2.

Proof of Corollary 3:
Let $u(x,t)$ be a nonnegative solution of (1) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ which satisfies $|u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. We introduce $w(y,s) = w_0(y,s)$ where w_0 is defined in (2). Then, it is easy to see that w satisfies all the hypotheses of Theorem 2. Therefore, either $w \equiv 0$ of there exists $t_0 \ge 0$ such that $\forall (y,s) \in \mathbb{R}^{N+1}$, $w(y,s) = \kappa(1 + t_0 e^s)^{-\frac{1}{p-1}}$. Thus, either $u \equiv 0$ or $u(x,t) = \kappa(T + t_0 - t)^{-\frac{1}{p-1}}$. This concludes the proof of Corollary 3.

A Proof of Proposition 3.2

We proceed in 3 steps: In Step 1, we give a new version of an ODE lemma by Filippas and Kohn [3] which will be applied in Step 2 in order to show that either v_{null} or v_+ is predominant in L^2_ρ as $s \to -\infty$. In Step 3, we show that in the case where v_+ is predominant, then either $v_0(s)$ or $v_1(s)$ predominates the other.

Step 1: An ODE lemma

Lemma A.1 *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions which are non negative and satisfy*
i) $(x,y,z)(s) \to 0$ as $s \to -\infty$, and $\forall s \le s_*$, $x(s) + y(s) + z(s) \ne 0$,
ii) $\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \le s_0$

$$\left\{ \begin{array}{rcl} \dot{z} & \ge & c_0 z - \epsilon(x+y) \\ |\dot{x}| & \le & \epsilon(x+y+z) \\ \dot{y} & \le & -c_0 y + \epsilon(x+z). \end{array} \right. \tag{57}$$

Then, either $x + y = o(z)$ or $y + z = o(x)$ as $s \to -\infty$.

Proof: Filippas and Kohn showed in [3] a slightly weaker version of this lemma (with in the conclusion $x, y, z \to 0$ exponentially fast instead of $x + y = o(z)$). We adapt here their proof to get the proof of lemma A.1.
By rescaling in time, we may assume $c_0 = 1$.

Part 1: Let $\epsilon > 0$. We show in this part that either:

$$\exists s_2(\epsilon) \text{ such that } \forall s \le s_2,\ z(s) + y(s) \le C\epsilon x(s), \tag{58}$$

$$\text{or } \exists s_2(\epsilon) \text{ such that } \forall s \le s_2,\ x(s) + y(s) \le C\epsilon z(s). \tag{59}$$

We first show that $\forall s \le s_0(\epsilon)$, $\beta(s) \le 0$ where $\beta = y - 2\epsilon(x+z)$.

We argue by contradiction and suppose that there exists $s_* \leq s_0(\epsilon)$ such that $\beta(s_*) > 0$. Then, if $s \leq s_*$ and $\beta(s) > 0$, we have form (57)
$\dot{\beta}(s) = \dot{y} - 2\epsilon(\dot{x} + \dot{z}) \leq -y + \epsilon(x+z) + 2\epsilon^2(x+y+z) - 2\epsilon(z - \epsilon(x+y)) \leq -\epsilon(1 - 4\epsilon - 8\epsilon^2)x - \epsilon(3 - 2\epsilon - 8\epsilon^2)z \leq 0$.
Therefore, $\forall s \leq s_*$, $\beta(s) \geq \beta(s_*) > 0$, which contradicts $\beta(s) \to 0$ as $s \to -\infty$. Thus

(60) $$\forall s \leq s_0(\epsilon),\ y \leq 2\epsilon(x+z).$$

Therefore, (57) yields

(61) $$\begin{array}{rcl} \dot{z} & \geq & \frac{1}{2}z - 2\epsilon x \\ |\dot{x}| & \leq & 2\epsilon(x+z) \end{array}$$

Let $\gamma(s) = 8\epsilon x(s) - z(s)$. Two cases arise then:
Case 1: $\exists s_2 \leq s_0(\epsilon)$ such that $\gamma(s_2) > 0$.
Suppose then $\gamma(s) = 0$ and compute $\dot{\gamma}(s)$.
$\dot{\gamma}(s) = 8\epsilon\dot{x} - \dot{z} \leq 16\epsilon^2(x+z) - \frac{1}{2}z + 2\epsilon x = -z(s)(\frac{1}{4} - 2\epsilon - 16\epsilon^2)$.
Since $z(s) > 0$ (otherwise $z(s) = 0$, $x(s) = 0$ and then $y(s) = 0$ by (60), which is excluded by the hypothesis), we have

$$\gamma(s) = 0 \Longrightarrow \dot{\gamma}(s) < 0.$$

Since $\gamma(s_2) > 0$, this implies $\forall s \leq s_2$, $\gamma(s) > 0$, i.e. $8\epsilon x(s) > z(s)$. Together with (60), this yields (58).

Case 2: $\forall s \leq s_0(\epsilon)$, $\gamma(s) \leq 0$ i.e. $8\epsilon x \leq z(s)$.
In this case, (61) yields

$$\forall s \leq s_0(\epsilon),\ \dot{z} \geq \frac{1}{4}z, \text{ and } \dot{x} \leq (2\epsilon + \frac{1}{4})z.$$

Therefore, we get by integration:

$$z(s) \geq \frac{1}{4}\int_{-\infty}^{s} z(t)dt \text{ and } x(s) \leq (2\epsilon + \frac{1}{4})\int_{-\infty}^{s} z(t)dt,$$

which yields $x(s) \leq (8\epsilon + 1)z(s)$. We inject this in (61) and get
$\dot{x}(s) \leq 2\epsilon(x+z) \leq 2\epsilon z(2 + 8\epsilon)$. Again, by integration:
$x(s) \leq 2\epsilon(2+8\epsilon)\int_{-\infty}^{s} z(t)dt \leq 8\epsilon(2+8\epsilon)z(s)$. Together with (60), this yields (59).

Part 2: Let $\epsilon < \frac{1}{C}$. Then either (58) or (59) occurs.
For example, (58) occurs, that is $\exists s_2(\epsilon) \leq s_0$ such that $\forall s \leq s_2$, $z + y \leq C\epsilon x$. Let $\epsilon' \leq \epsilon$ be an arbitrary positive number. Then, according to Part 1, either $\forall s \leq s_2'$, $z + y \leq C\epsilon' x$ for some $s_2'(\epsilon')$,
or $\forall s \leq s_2'$, $y + x \leq C\epsilon' z$ for some $s_2'(\epsilon')$.
Only the first case occurs. Indeed, if not, then for $s \leq \min(s_2, s_2')$, $x \leq C\epsilon' z \leq C\epsilon' C\epsilon x \leq C^2\epsilon^2 x$ since $\epsilon' \leq \epsilon$. Since $(C\epsilon)^2 < 1$, we have $x \equiv 0$ and $z \equiv y \equiv 0$ for $s \leq \min(s_2, s_2')$, which is excluded by the hypotheses.
Do the same if (59) occurs.
This concludes the proof of lemma A.1. ■

Step 2: Competition between v_+, v_{null} and v_-
In this step we show that either $\|v_-(s)\|_{L^2_\rho} + \|v_+(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$ (which is case *iii*) of Proposition 3.2) or

$\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$ (which yields case *i)* or *ii)* of Proposition 3.2) in Step 3).

This situation is exactly symmetric to the one in section 4 in Filippas and Kohn's paper [3]. Indeed, we are treating the same equation (38), but we have $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$ whereas in [3], $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to +\infty$. Nevertheless, the derivation of the differential inequalities satisfied by v_-, v_{null} and v_+ in [3] is still valid here with the changes: "$s \to +\infty$" becomes $s \to -\infty$ and "s large enough" becomes "$-s$ large enough". Therefore, we claim that [3] implies:

Lemma A.2 *$\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that for a.e. $s \le s_0$:*

$$\begin{aligned} \dot{z} &\ge (\tfrac{1}{2} - \epsilon) z - \epsilon(x+y) \\ |\dot{x}| &\le \epsilon(x+y+z) \\ \dot{y} &\le -(\tfrac{1}{2} - \epsilon) y + \epsilon(x+z) \end{aligned}$$

where $z(s) = \|v_+(s)\|_{L^2_\rho}$, $x(s) = \|v_{null}(s)\|_{L^2_\rho}$ and
$y(s) = \|v_-(s)\|_{L^2_\rho} + \||y|^{\frac{k}{2}} v^2(s)\|_{L^2_\rho}$ for a fixed integer k.

Now, since $\|v(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$, we have $(x,y,z)(s) \to 0$ as $s \to -\infty$. We can not have $x(s_1) + y(s_1) + z(s_1) = 0$ for some $s_1 \in \mathbb{R}$, because this implies that $\forall y \in \mathbb{R}^N$, $v(y, s_1) = 0$, and from the uniqueness of the solution to the Cauchy problem of equation (38) and $v(s_1) = 0$, we have $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $v(y,s) = 0$, which contradicts $\kappa + v \to 0$ as $s \to +\infty$. Applying lemma A.1 with $c_0 = \frac{1}{4}$, we get:
either $\|v_-(s)\|_{L^2_\rho} + \|v_+(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$
or $\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$.

Step 3: Competition between v_0 and v_1

In this step, we focus on the case where $\|v_-(s)\|_{L^2_\rho} + \|v_{null}(s)\|_{L^2_\rho} = o(\|v_+(s)\|_{L^2_\rho})$. We will show that it leads either to case *i)* or case *ii)* of Proposition 3.2.

Let us first remark that lemma A.1 implies in this case that

$$\forall \epsilon > 0,\ z(s) = \|v_+(s)\|_{L^2_\rho} = O(e^{(\frac{1}{2} - \epsilon)}) \text{ as } s \to -\infty. \tag{62}$$

Now, we want to derive from (38) the equations satisfied by v_0 and v_1. We must estimate $\int f(v(y,s)) k_m(y_i) \rho(y) dy$ for $m = 0, 1$ and $i = 1, ..N$ (see (41) for k_m). Let us give this crucial estimate:

Lemma A.3 *There exists $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \le s_0$, $\int v^2 |y|^{k'} \rho dy \le c_0(k') \delta^{4-k'} z(s)^2$.*

Proof: Let $I(s) = \left(\int v^2 |y|^{k'} \rho dy\right)^{1/2}$. We first derive a differential inequality satisfied by $I(s)$. If we multiply (38) by $v|y|^{k'}\rho$ and integrate over $\mathbb{R}^N$, we obtain:

$$\frac{1}{2} \frac{d}{ds} (I(s)^2) = \int v.\mathcal{L}v |y|^{k'} \rho dy + \int v f(v) |y|^{k'} \rho dy.$$

Since v is bounded by M, we get $\int v f(v) |y|^{k'} \rho dy \le MC \int v^2 |y|^{k'} \rho dy$.

After some calculations, we show that
$\int v.\mathcal{L}v |y|^{k'} \rho dy \le \frac{k}{2}(k+N-2) \int |y|^{k'-2} v^2 \rho dy + (1 - \frac{k}{4}) I(s)^2$.

Using Schwartz's inequality, we find:
$\int v^2|y|^{k'-2}\rho dy \leq I(s)\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2}$.
Let us bound $\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2}$. If $k'>4$ and $\delta>0$, then
$\left(\int v^2|y|^{k'-4}\rho dy\right)^{1/2} \leq \left(\int_{|y|\leq\delta^{-1}} v^2|y|^{k'-4}\rho dy\right)^{1/2} + \left(\int_{|y|\geq\delta^{-1}} v^2|y|^{k'-4}\rho dy\right)^{1/2}$
$\leq \delta^{2-k'/2}\left(\int v^2\rho dy\right)^{1/2} + \delta^2 I$
$\leq 2\delta^{2-k'/2}z(s) + \delta^2 I$ since $\left(\int v^2\rho dy\right)^{1/2} \sim \left(\int v_+^2\rho dy\right)^{1/2} = z(s)$ as $s\to-\infty$.

Combining all the previous bounds, we obtain:
$I'(s) \leq -\theta I + d\delta^{2-k'/2}z$ with $\theta = \frac{k'}{4} - 1 - MC - \frac{k'}{2}(k'+N-2)\delta^2$ and $d = k'(k'+N-2)$.

We claim that there exist an integer $k'>4$ and $\delta_0>0$ such that $\forall\delta\in(0,\delta_0)$, $\theta\geq 1$. Hence,

$$I'(s) \leq -I(s) + d\delta^{2-k'/2}z(s). \tag{63}$$

Now, we will derive a differential inequality satisfied by z in order to couple it with (63), and then prove lemma A.3.

We project (38) onto the positive subspace of $\mathcal{L}$, we multiply the result by $v_+\rho$ and then, we integrate over $\mathbb{R}^N$ to get:

$$\frac{1}{2}\frac{d}{ds}(z(s)^2) = \int \mathcal{L}v_+.v_+\rho dy + \int P_+(f(v))v_+\rho dy.$$

Since $(\text{Spec } \mathcal{L})\cap\mathbb{R}_+^* = \{1,\frac{1}{2}\}$, we have $\int \mathcal{L}v_+.v_+\rho dy \geq \frac{1}{2}z(s)^2$.
Using Schwartz's inequality, we obtain:
$\left|\int P_+(f(v))v_+\rho dy\right| \leq \left(\int P_+(f(v))^2\rho dy\right)^{1/2}\left(\int v_+^2\rho dy\right)^{1/2}$
$\leq \left(\int f(v)^2\rho dy\right)^{1/2} z(s)$.
Since $v\to 0$ as $s\to-\infty$ uniformly on compact sets, we have:
$\int f(v)^2\rho dy \leq C^2\int v^4\rho dy = C^2\int_{|y|\leq\delta^{-1}} v^4\rho dy + C^2\int_{|y|\geq\delta^{-1}} v^4\rho dy$
$\leq \epsilon^2\int v^2\rho dy + C^2M^2\delta^{k'}\int v^2|y|^{k'}\rho \leq 4\epsilon^2z^2 + C^2M^2\delta^{k'}I^2$ for all $\epsilon>0$, provided that $s\leq s_0(\epsilon,\delta)$.
Thus, $\left(\int f(v)^2\rho dy\right)^{1/2} \leq 2\epsilon z + CM\delta^{k'/2}I$.

Combining all the previous estimates, we obtain:

$$z'(s) \geq \frac{1}{2}z(s) - 2\epsilon z - CM\delta^{k'/2}I(s). \tag{64}$$

With $\epsilon = 1/8$, (63) and (64) yield:

$$\forall s\leq s_0 \left\{\begin{array}{lcl} z'(s) & \geq & \frac{1}{4}z(s) - CM\delta^{k'/2}I(s) \\ I'(s) & \leq & -I(s) + d\delta^{2-k'/2}z(s). \end{array}\right.$$

Now, we are ready to conclude the proof of lemma A.3:

Let $\gamma(s) = I(s) - 2d\delta^{2-k'/2}z(s)$. Let us assume $\gamma(s)>0$ and show that $\gamma'(s)<0$.

$\gamma'(s) = I' - 2d\delta^{2-k'/2}z' \leq (-I + d\delta^{2-k'/2}z) - 2d\delta^{2-k'/2}(\frac{1}{4}z - CM\delta^{k'/2}I)$
$\leq I(-1+\frac{1}{4}+2CMd\delta^2) = I(-\frac{3}{4}+2CM\delta^2 d)$

If we choose δ_0 such that $\forall\delta\in(0,\delta_0)$, $-\frac{3}{4}+2CM\delta^2 d<0$, then $\gamma(s)>0$ implies $I(s)>0$ and $\gamma'(s)<0$. Since $\gamma(s)\to 0$ as $s\to-\infty$ (because $v\to 0$

uniformly on compact sets), we conclude that for some $s_1 \in \mathbb{R}$, $\forall s \leq s_1$, $\gamma(s) \leq 0$. Since $d = k'(k'+N-2)$, lemma A.3 is proved. ■

Using lemma A.3, we try to estimate $\int f(v) k_m(y_i) \rho dy$.
Since $|f(v) - \frac{p}{2\kappa} v^2| \leq C(M) v^3$, we write:

$$\int f(v) \rho dy = \frac{p}{2\kappa} \int v^2 \rho dy + O(\int v^3 \rho dy). \tag{65}$$

For all $\epsilon > 0$, $\delta > 0$ and $s \leq s_0$, we write:
$|\int v^3 \rho dy| \leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + |\int_{|y| \geq \delta^{-1}} v^3 \rho dy|$
$\leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + M\delta^{k'} \int v^2 |y|^{k'} \rho dy \leq |\int_{|y| \leq \delta^{-1}} v^3 \rho dy| + Mc_0(k')\delta^4 z(s)^2$.
We fix $\delta > 0$ small enough such that $Mc_0(k')\delta^4 \leq \frac{\epsilon}{2}$. Then, we take $s \leq s_1(\epsilon)$ such that $|\int_{|y| \leq \delta^{-1}} v^3 \rho dy| \leq \frac{\epsilon}{4} \int_{|y| \leq \delta^{-1}} v^2 \rho dy \leq \frac{\epsilon}{4} \int v^2 \rho dy$ (because $v \to 0$ in $L^\infty(B(0,\delta))$).

Since $\int v^2 \rho dy \sim z(s)^2$ as $s \to -\infty$, we get for $s \leq s_2(\epsilon)$, $|\int v^3 \rho dy| \leq \epsilon z(s)^2$. Therefore, equation (38) and (65) yield:

$$v_0'(s) = v_0(s) + \frac{p}{2\kappa} z(s)^2 (1 + \alpha(s)) \tag{66}$$

where $\alpha(s) \to 0$ as $s \to -\infty$.

Using the same type of calculations as for $\int v^3 \rho dy$, we can prove that $\int v^2 k_1(y_i) \rho dy = O(z(s)^2)$. Therefore, (38) yields the following vectorial equation:

$$v_1'(s) = \frac{1}{2} v_1(s) + \beta(s) z(s)^2 \tag{67}$$

where β is bounded.

From (62), (66), (67) and standard ODE techniques, we get:

$$\forall \epsilon > 0,\ v_0(s) = O(e^{(1-\epsilon)s}) \text{ and } v_1(s) = C_1 e^{\frac{s}{2}} + O(e^{(1-\epsilon)s}).$$

Since $z(s)^2 = v_0(s)^2 + 2|v_1(s)|^2$, we write (66) as

$$v_0'(s) = v_0(s) + \frac{p}{2\kappa} |C_1|^2 e^s (1 + \alpha(s)) + \gamma(s)$$

where $\gamma(s) = O(e^{2(1-\epsilon)s})$. Therefore,

$$\forall \epsilon > 0,\ v_0(s) = \frac{p}{\kappa} |C_1|^2 s e^s (1 + o(1)) + C_0 e^s + O(e^{2(1-\epsilon)s}) \tag{68}$$

as $s \to -\infty$.

Two cases arise:

i) If $C_1 \neq 0$, then $v_1(s) \sim C_1 e^{\frac{s}{2}} \gg \frac{p}{\kappa} |C_1|^2 s e^s \sim v_0(s)$. This is case *ii)* of Proposition 3.2.

ii) If $C_1 = 0$, then $|z(s)| \leq C e^{(1-\epsilon)s}$, and (67) yields $v_1 = O(e^{(2-\epsilon)s})$. From (68), we have $v_0(s) = C_0 e^s + O(e^{(2-\epsilon)s})$.
We claim that $C_0 < 0$ (If not, then the function $F(s) = e^{-s} v_0(s)$ goes to $C_0 \geq 0$ as $s \to -\infty$ and is increasing if $s \leq s_0$. Therefore, $\forall s \leq s_0$, $v_0(s) \geq C_0 e^s \geq 0$. Since v is bounded and $\kappa + v \geq 0$, we have from Proposition 3.5 $\forall s \in \mathbb{R}$, $\int (\kappa + v(y,s)) \rho dy \leq \kappa$, that is $v_0(s) \leq 0$.

Hence, $\forall s \le s_0$, $v_0(s) = 0$ and $z(s) = \sqrt{2}|v_1(s)|$. Then, (67) implies that $\forall s \le s_0$, $v_1(s) = 0$ and $z(s) = 0$. Since $\int v^2 \rho dy \sim z(s)$, we have $v \equiv 0$ and $w \equiv \kappa$ in a neighborhood of $-\infty$ and then on $\mathbb{R}^N \times \mathbb{R}$ which contradicts $w \to 0$ as $s \to +\infty$).

Thus, $v_0(s) \sim C_0 e^s \gg Ce^{(2-\epsilon)s} \ge |v_1(s)|$. This is Case i) of Proposition 3.2.

This concludes the proof of Proposition 3.2. ■

B Proof of Proposition 3.4

Let us recall Proposition 3.4:

Proposition B.1 *Let V be an L^∞ solution of*

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V) \tag{69}$$

defined for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $l(s) = -\frac{pe^s}{(p-1)(1+e^s)}$ and $F(V) = |\varphi + V|^{p-1}(\varphi + V) - \varphi^p - p\varphi^{p-1}V$.

Assume that $V \to 0$ as $s \to \pm\infty$ uniformly on compact sets of $\mathbb{R}^N$,

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} = o(e^s) \text{ as } s \to -\infty. \tag{70}$$

Then $V \equiv 0$.

In order to show that $V \equiv 0$ in $\mathbb{R}^{N+1}$, we proceed in three steps: in Step 1, we do an L^2_ρ analysis for V as $s \to -\infty$, similarly as in Part 2 of section 2 to show that either $\|V(s)\|_{L^2_\rho} \sim \|V_+(s)\|_{L^2_\rho}$ or $\|V(s)\|_{L^2_\rho} \sim \|V_{null}(s)\|_{L^2_\rho}$. Then, we treat these two cases successively in Steps 2 and 3 to show that $V \equiv 0$.

Step 1: L^2_ρ analysis for V as $s \to -\infty$

Lemma B.1 *As $s \to -\infty$, either*
i) $\|V_-(s)\|_{L^2_\rho} + \|V_{null}(s)\|_{L^2_\rho} = o(\|V_+(s)\|_{L^2_\rho})$
or ii) $\|V_-(s)\|_{L^2_\rho} + \|V_+(s)\|_{L^2_\rho} = o(\|V_{null}(s)\|_{L^2_\rho})$.

Proof: One can adapt easily the proof of Filippas and Kohn in [3] here. Indeed, V satisfies almost the same type of equation (because $l(s) \to 0$ as $s \to -\infty$, and $|F(V)| \le CV^2$), and $V \to 0$ as $s \to -\infty$ uniformly on compact sets. Therefore, we claim that up to the change of "$s \to -\infty$ " into "$s \to +\infty$", section 4 of [3] implies

Lemma B.2 *$\forall \epsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that for a.e. $s \le s_0$:*

$$\begin{aligned} \dot{Z} &\ge (\tfrac{1}{2} - \epsilon)Z - \epsilon(X+Y) \\ |\dot{X}| &\le \epsilon(X+Y+Z) \\ \dot{Y} &\le -(\tfrac{1}{2} - \epsilon)Y + \epsilon(X+Z) \end{aligned}$$

where $Z(s) = \|V_+(s)\|_{L^2_\rho}$, $X(s) = \|V_{null}(s)\|_{L^2_\rho}$ and
$Y(s) = \|V_-(s)\|_{L^2_\rho} + \||y|^{\frac{k}{2}}V^2(s)\|_{L^2_\rho}$ for a fixed integer k.

Since $\|V(s)\|_{L^\infty_{loc}} \to 0$ as $s \to -\infty$ and V is bounded in L^∞, we have $(X,Y,Z)(s) \to 0$ as $s \to -\infty$. Similarly as in Step 2 of Appendix A, we can not have $X(s)+Y(s)+Z(s)=0$ for some $s \in \mathbb{R}$. Therefore, the conclusion follows from lemma A.1, in the same way as in Step 2 of Appendix A.

Step 2: Case $\|V_-(s)\|_{L^2_\rho} + \|V_{null}(s)\|_{L^2_\rho} = o(\|V_+(s)\|_{L^2_\rho})$

Since (69) and (38) are very similar (the only real difference is the presence in (69) of $l(s)$ which goes to zero as $s \to -\infty$), one can adapt without difficulty all the Step 3 of Appendix A and show that V_0 and V_1 satisfy equations analogous to (66) and (67): $\forall s \le s_0$

$$\left\{ \begin{array}{lcl} V_0'(s) & = & V_0(s)(1+l(s)) + a_0(s)(V_0(s)^2 + 2|V_1(s)|^2) \\ V_1'(s) & = & V_1(s)(\frac{1}{2}+l(s)) + a_1(s)(V_0(s)^2 + 2|V_1(s)|^2) \end{array} \right. \tag{71}$$

where a_0 and a_1 are bounded.

According to (70), there exist $B>0$ and $s_1 \le s_0$ such that $\forall s \le s_1$

$$|a_0(s)| \le B, |a_1(s)| \le B, |V_0(s)| \le e^{\frac{3s}{2}} \text{and } |V_1(s)| \le e^{\frac{3s}{2}}. \tag{72}$$

We claim then that the following lemma yields $V \equiv 0$:

Lemma B.3 *$\forall n \in \mathbb{N}$, $\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^n-1}e^{3\times 2^{n-1}s}$ for $m=0$ and $m=1$, where $e(s_1) = e^{-\int_{-\infty}^{s_1} l(t)dt}$.*

Indeed, the lemma yields that $\forall s \le s_2$ $V_0(s) = V_1(s) = 0$ for some $s_2 \le s_1$. Since $\|V(s)\|_{L^2_\rho} \sim \|V_+(s)\|_{L^2_\rho}$ as $s \to -\infty$, we have $\forall s \le s_3$, $\forall y \in \mathbb{R}^N$, $V(y,s) = 0$ for some $s_3 \le s_2$. The uniqueness of the solution of the Cauchy problem: $\forall s \ge s_3$, V satisfies equation (69) and $V(s_3) = 0$ yields $V \equiv 0$ in $\mathbb{R}^{N+1}$.

Proof of lemma B.3: We proceed by induction:

- $n=0$, the hypothesis is true by (72).

- We suppose that for $n \in \mathbb{N}$, we have

$\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^n-1}e^{3\times 2^{n-1}s}$ for $m=0,1$. Let us prove that $\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{3\times 2^n s}$ for $m=0,1$.

Let $F_m(s) = V_m(s)e^{-(1-\frac{m}{2})s - \int_{-\infty}^{s} l(t)dt}$. From (71) and the induction hypothesis, we have: $\forall s \le s_1$,

$|F_m'(s)| \le e^{-(1-\frac{m}{2})s - \int_{-\infty}^{s_1} l(t)dt} B \times 3(\frac{3}{2}e(s_1)B)^{2(2^n-1)}e^{3\times 2^n s}$. By the induction hypothesis, $\lim_{s\to-\infty} F_m(s) = 0$. Hence, $\forall s \le s_1$,

$|F_m(s)| = |\int_{-\infty}^{s} F_m'(\sigma)d\sigma| \le \int_{-\infty}^{s} |F_m'(\sigma)|d\sigma$

$\le 3e(s_1)B(\frac{3}{2}e(s_1)B)^{2^{n+1}-2} \int_{-\infty}^{s} e^{(3\times 2^n - (1-\frac{m}{2}))\sigma}d\sigma$

$= \frac{2}{3\times 2^n - (1-\frac{m}{2})}(\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{(3\times 2^n-(1-\frac{m}{2}))s}$.

Since $3\times 2^n - (1-\frac{m}{2}) \ge 2$ and $l(s) \le 0$, this yields

$\forall s \le s_1$, $|V_m(s)| \le (\frac{3}{2}e(s_1)B)^{2^{n+1}-1}e^{3\times 2^n s}$ for $m=0,1$. This concludes the proof of lemma B.3. ■

Step 3: Case $\|V_-(s)\|_{L^2_\rho} + \|V_+(s)\|_{L^2_\rho} = o(\|V_{null}(s)\|_{L^2_\rho})$

In order to show that $V \equiv 0$, it is enough to show that $V_{null} \equiv 0$ or equivalently that $\forall i,j \in \{1,..,N\}$, $V_{2,ij} \equiv 0$.

For this purpose, we derive form (69) an equation satisfied by $V_{2,ij}$ as $s \to -\infty$:

(73) $$V'_{2,ij}(s) = l(s)V_{2,ij}(s) + \int F(V)\frac{H_{2,ij}}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho dy.$$

We have to estimate the last term of (73):
- if $i = j$, then $H_{2,ij}(y) = y_i^2 - 2$ and

(74) $$|\int F(V)H_{2,ii}\rho dy| \leq C\int V^2\rho dy + C\int V^2|y|^2\rho dy,$$

- if $i \neq j$, then $H_{2,ij}(y) = y_iy_j$ and

(75) $$|\int F(V)H_{2,ij}\rho dy| \leq C\int V^2|y|^2\rho dy.$$

The hypothesis of this step implies that

(76) $$\int V^2\rho dy \leq 2\int V^2_{null}\rho dy.$$

It remains then to bound $\int V^2|y|^2\rho dy$. This will be done through this lemma, which is analogous to lemma A.3:

Lemma B.4 *There exists $\delta_0 > 0$ and an integer $k' > 5$ such that for all $\delta \in (0,\delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int V^2|y|^{k'}\rho dy \leq c_0(k')\delta^{4-k'}\int V^2_{null}\rho dy$.*

Proof: We will argue similarly as in the proof of lemma A.3.
Let $I(s) = \left(\int V^2|y|^{k'}\rho dy\right)^{1/2}$ and use the notation $X(s) = \left(\int V^2_{null}|y|^{k'}\rho dy\right)^{1/2}$.
From (69), we derive the following equation for $I(s)$:

$$\frac{1}{2}\frac{d}{ds}(I(s)^2) = \int V.\mathcal{L}V|y|^{k'}\rho dy + l(s)I(s)^2 + \int VF(V)|y|^{k'}\rho dy.$$

Since v is bounded by M, we can assume $|V| \leq M + 1 = M'$ and get $\int VF(V)|y|^{k'}\rho dy \leq M'C\int V^2|y|^{k'}\rho dy$. We can also assume that $|l(s)| \leq \frac{1}{12}$.

As for lemma A.3, we can show that for all $\delta > 0$ $\int V.\mathcal{L}V|y|^{k'}\rho dy \leq \frac{k'}{2}(k'+N-2)I(s)(\delta^{2-k'/2}\left(\int V^2\rho dy\right)^{1/2} + \delta^2 I) + (1-\frac{k}{4})I(s)^2$.

Combining these bounds with (76), we get:
$I'(s) \leq -\theta I + d\delta^{2-k'/2}z$ with $\theta = \frac{k'}{4} - 1 - \frac{1}{12} - M'C - \frac{k'}{2}(k'+N-2)\delta^2$ and $d = k'(k'+N-2)$.

It is clear that there exist an integer $k' > 5$ and $\delta_0 > 0$ such that $\forall\delta \in (0,\delta_0)$, $\theta \geq 1$. Hence,

(77) $$I'(s) \leq -I(s) + d\delta^{2-k'/2}X(s).$$

Let us derive a differential equation satisfied by X.
From (69), we obtain:

$$\frac{1}{2}\frac{d}{ds}(X(s)^2) = l(s)X(s)^2 + \int P_{null}(F(V))V_{null}\rho dy.$$

By Schwartz's inequality, we have:
$\left|\int P_{null}(F(V))V_{null}\rho dy\right| \leq \left(\int P_{null}(F(V))^2\rho dy\right)^{1/2}\left(\int V^2_{null}\rho dy\right)^{1/2}$

$\leq \left(\int F(V)^2\rho dy\right)^{1/2} X(s)$.
Since $V \to 0$ as $s \to -\infty$ uniformly on compact sets, we have:
$\int F(V)^2\rho dy \leq C^2 \int V^4 \rho dy = C^2 \int_{|y|\leq \delta^{-1}} V^4\rho dy + C^2 \int_{|y|\geq\delta^{-1}} V^4 \rho dy$
$\leq \epsilon^2 \int V^2\rho dy + C^2 M'^2 \delta^{k'} \int V^2 |y|^{k'}\rho \leq 4\epsilon^2 X^2 + C^2 M'^2 \delta^{k'} I^2$ for all $\epsilon > 0$, provided that $s \leq s_0(\epsilon, \delta)$.
Thus, $\left(\int F(V)^2 \rho dy\right)^{1/2} \leq 2\epsilon X + CM'\delta^{k'/2} I$.

Since $|l(s)| \leq \frac{1}{12}$, we combine all the previous bounds to get:

$$|X'(s)| \leq (2\epsilon + \frac{1}{12})X(s) + CM'\delta^{k'/2} I(s). \tag{78}$$

With $\epsilon = 1/12$, (77) and (78) yield:

$$\forall s \leq s_1 \left\{ \begin{array}{rcl} |X'(s)| & \leq & \frac{1}{4}X(s) + CM'\delta^{k'/2}I(s) \\ I'(s) & \leq & -I(s) + d\delta^{2-k'/2}X(s). \end{array} \right.$$

Now, we conclude the proof of lemma A.3:

Let $\gamma(s) = I(s) - 2d\delta^{2-k'/2}X(s)$. Let us assume $\gamma(s) > 0$ and show that $\gamma'(s) < 0$.

$\gamma'(s) = I' - 2d\delta^{2-k'/2}X'$
$\leq (-I + d\delta^{2-k'/2}X) + 2d\delta^{2-k'/2}(\frac{1}{4}X(s) + CM'\delta^{k'/2}I)$
$\leq I(-1 + \frac{1}{2} + 2CM'd\delta^2 + \frac{1}{4}) = I(-\frac{1}{4} + 2CM'\delta^2 d)$

If we choose δ_0 such that $\forall \delta \in (0, \delta_0)$, $-\frac{1}{4} + 2CM'\delta^2 d < 0$, then $\gamma(s) > 0$ implies $I(s) > 0$ and $\gamma'(s) < 0$. Since $\gamma(s) \to 0$ as $s \to -\infty$ (because $V \to 0$ uniformly on compact sets), we conclude that for some $s_2 \in \mathbb{R}$, $\forall s \leq s_1$, $\gamma(s) \leq 0$. Since $d = k'(k' + N - 2)$, lemma B.4 is proved. ■

Lemma B.4 allows us to bound $\int V^2|y|^2\rho dy$. Indeed, for fixed $\delta \in (0, \delta_0)$ and $s \leq s_0$, we have:
$\int V^2|y|^2\rho dy \leq \int_{|y|\leq\delta^{-1}} V^2|y|^2\rho dy + \int_{|y|\geq\delta^{-1}} V^2|y|^2\rho dy$
$\leq \delta^{-2}\int_{|y|\leq\delta^{-1}} V^2\rho dy + \delta^{k'-2}\int_{|y|\geq\delta^{-1}} V^2|y|^{k'}\rho dy$
$\leq \delta^{-2}\int V^2\rho dy + c_0(k')\delta^2\int V^2\rho dy = C(\delta, k')\int V^2\rho dy$.

With this bound, (74) and (75), equation (69) yields: $\forall s \leq s_0$,

$$V'_{2,ij}(s) = l(s)V_{2,ij}(s) + a_{2,ij}(s)\|V_{null}(s)\|^2_{L^2_\rho}$$

where $a_{2,ij}$ is bounded.

According to (70), there exist then $B > 0$ and $s_1 \leq s_0$ such that $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$,

$$|a_{2,ij}(s)| \leq B,\ |V_{2,ij}(s)| \leq e^s.$$

We claim that the following lemma yields $V \equiv 0$:

Lemma B.5 $\forall n \in \mathbb{N}$, $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$,
$|V_{2,ij}(s)| \leq (8N^2(N+1)^2 e(s_1)B)^{2^n-1} e^{2^n s}$ *where* $e(s_1) = e^{-\int_\infty^{s_1} l(t)dt}$.

Indeed, this lemma yields $\forall s \leq s_1$, $\forall i, j \in \{1, ..., N\}$, $V_{2,ij}(s) = 0$ for some $s_2 \leq s_1$. Hence, $\forall s \leq s_2$, $\forall y \in \mathbb{R}^N$, $V_{null}(y, s) = 0$, and by the hypothesis of this step, $\forall s \leq s_3$, $\forall y \in \mathbb{R}^N$, $V(y, s) = 0$ for some $s_3 \leq s_3$. The uniqueness of the solutions to the Cauchy problem: $\forall s \geq s_3$, V satisfies equation (69) and $V(s_3) = 0$ yields $V \equiv 0$ in $\mathbb{R}^{N+1}$.

We escape the proof of lemma B.5 since it is completely analogous to the proof of lemma B.3.

C Proof of $i)$ of Proposition 3.7

We proceed in 4 steps: in Step 1, we derive form the fact that $\|v(s)\|_{L^2_\rho} \sim \|v_{null}(s)\|_{L^2_\rho}$ an equation satisfied by $v_{null}(s)$ as $s \to -\infty$. Then, we find in Step 2 $c > 0$, $C > 0$ and $s_0 \in \mathbb{R}$ such that $c|s|^{-1} \leq \|v(s)\|_{L^2_\rho} \leq C|s|^{-1}$ for $s \leq s_0$. In Step 3, we use this estimate to derive a more accurate equation for v_{null}. We use this equation in Step 4 to get the asymptotic behaviors of $v_{null}(y,s)$, $v_0(s)$ and $v_1(s)$.

Step 1: An ODE satisfied by $v_{null}(y,s)$ as $s \to -\infty$
This step is very similar to Step 3 in Appendix B where we handled the equation (69) instead of (38) as in the present context.
From (38) we have by projection:

$$v'_{2,ij}(s) = \int f(v) \frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}} \rho(y) dy \tag{79}$$

We will prove the following proposition here:

Proposition C.1 *i)* $\forall i,j \in \{1,...,N\}$,

$$v'_{2,ij}(s) = \frac{p}{2\kappa} \int v^2_{null}(y,s) \frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}} \rho(y) dy + o(\|v_{null}(s)\|^2_{L^2_\rho}). \tag{80}$$

as $s \to -\infty$ *.*

ii) There exists a symmetric $N \times N$ *matrix* $A(s)$ *such that* $\forall s \in \mathbb{R}$,

$$v_{null}(y,s) = y^T A(s) y - 2tr(A(s)) \tag{81}$$

$$\text{and } c_0\|A(s)\| \leq \|v_{null}(s)\|_{L^2_\rho} \leq C_0\|A(s)\| \tag{82}$$

for some positive constants c_0 *and* C_0*. Moreover,*

$$A'(s) = \frac{4p}{\kappa} A^2(s) + o(\|A(s)\|^2) \text{ as } s \to -\infty. \tag{83}$$

Remark: $\|A\|$ stands for any norm on the space of $N \times N$ symmetric matrices.
Remark: The interest of the introduction of the matrix $A(s)$ is that it generalizes to $N \geq 2$ the situation of $N = 1$. Indeed, if $N = 1$, then it is obvious that $v_{null}(y,s) = yv_2(s)y - 2v_2(s)$ and that (80) implies $v'_2(s) = \frac{4p}{\kappa} v_2(s)^2 + o(v_2(s)^2)$. Let us remark that in the case $N = 1$, we get immediately $v_2(s) \sim -\frac{\kappa}{4ps}$ as $s \to -\infty$, which concludes the proof of Proposition 3.7. Unfortunately, we can not solve the system (83) so easily if $N \geq 2$. Nevertheless, the intuition given by the case $N = 1$ will guide us in next steps in order to refine the system (83) and reach then a similar result (see Step 2).

Proof of Proposition C.1:

Let us remark that $ii)$ follows directly form $i)$. Indeed, we have by definition of $H_{2,ij}$ and $v_{2,ij}$ (see (44) and (45)):
$v_{null}(y,s) = \sum_{i \leq j} v_{2,ij}(s) H_{2,ij}(y) = \sum_{i=1}^{N} v_{2,ii}(s)(y_i^2 - 2) + \sum_{i<j} v_{2,ij}(s) y_i y_j$. If we define $A(s) = (a_{ij}(s))_{i,j}$ by

$$a_{ii}(s) = v_{2,ii}(s), \text{ and for } i < j, \ a_{ij}(s) = a_{ji}(s) = \frac{1}{2} v_{2,ij}(s), \tag{84}$$

then (81) follows. (82) follows form the equivalence of norms in finite dimension $\frac{N(N+1)}{2}$. (83) follows from (80) by simple but long calculations which we escape here.

Now, we focus on the proof of i). For this purpose, we try to estimate the right-hand side of equation (79).

As in Step 3 of Part 3, this will be possible thanks to the following lemma:

Lemma C.1 *There exists $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int v^2 |y|^{k'} \rho dy \leq c_0(k') \delta^{4-k'} \int v_{null}^2 \rho dy$.*

Proof: The proof of lemma B.4 holds for lemma C.1 with the changes $V \to v$, $F \to f$ and $l(s) \to 0$. ∎

Now we estimate $\int f(v) H_{2,ij} \rho dy$:

Since $f(v) = \frac{p}{2\kappa} v^2 + g(v)$ where $|g(v)| \leq C|v|^3$, we write:

$$\int f(v) H_{2,ij} \rho dy = \frac{p}{2\kappa} \int v_{null}^2 H_{2,ij} \rho dy + (I) + (II) \tag{85}$$

where

$$(I) = \frac{p}{2\kappa} \int (v^2 - v_{null}^2) H_{2,ij} \rho dy \tag{86}$$

$$\text{and } (II) = \int g(v) H_{2,ij} \rho dy. \tag{87}$$

The proof of Proposition C.1 will be complete if we show that (I) and (II) are $o(\|v_{null}(s)\|_{L^2_\rho})$. Since $H_{2,ij}(y) = y_i^2 - 2$ if $i = j$ and $H_{2,ij}(y) = y_i y_j$ if $i \neq j$, it is enough to show that for all $\epsilon > 0$, I_1, I_2, II_1 and II_2 are lower that $\epsilon \|v_{null}(s)\|_{L^2_\rho}$ for all $s \leq s_0(\epsilon)$, where

$$\begin{aligned} I_1 &= \int |v^2 - v_{null}^2| \rho dy, & I_2 &= \int |v^2 - v_{null}^2| |y|^2 \rho dy, \\ II_1 &= \int |g(v)| \rho dy, & II_2 &= \int |g(v)| |y|^2 \rho dy. \end{aligned}$$

We start with I_1: Since $\int v^2 \rho dy \sim \int v_{null}^2 \rho dy$,
$I_1 = \int (v_+^2 + v_-^2) \rho dy \leq \epsilon \int v_{null}^2 \rho dy$ if $s \leq s_1(\epsilon)$.

For I_2, we consider $\delta \in (0, \delta_0)$, and write:
$I_2 \leq \int_{|y| \leq \delta^{-1}} |v^2 - v_{null}^2| |y|^2 \rho dy + \int_{|y| \geq \delta^{-1}} |v^2 - v_{null}^2| |y|^2 \rho dy :\ = I_{21} + I_{22}$.
We first estimate I_{21}:

Since $v = v_- + v_{null} + v_+$, we have $v^2 - v_{null}^2 = (v_+ + v_-)^2 + 2 v_{null}(v_+ + v_-)$.
Hence,
$I_{21} \leq \int_{|y| \leq \delta^{-1}} (v_+ + v_-)^2 |y|^2 \rho dy + 2 \int_{|y| \leq \delta^{-1}} |v_{null}(v_+ + v_-)| |y|^2 \rho dy$
$\leq \delta^{-2} \int (v_+ + v_-)^2 \rho dy + 2 \left(\int v_{null}^2 |y|^4 \rho dy\right)^{1/2} \left(\int (v_+ + v_-)^2 \rho dy\right)^{1/2}$.
Since $\int v^2 \rho dy \sim \int \int v_{null}^2 \rho dy$, we have
$\int (v_+ + v_-)^2 \leq \delta^3 \int v_{null}^2 \rho dy$ if $s \leq s_2(\delta)$.
Since the null subspace of $\mathcal{L}$ in finite dimensional, all the norms on it are equivalent, therefore, there exists $C_4(N)$ such that:
$\int v_{null}^2 |y|^4 \rho dy \leq C_4(N)^2 \int v_{null}^2 \rho dy$.
Therefore, $I_{21} \leq (\delta + 2C_4(N)\delta^{3/2}) \int v_{null}^2 \rho dy$ if $s \leq s_2(\delta)$.

For I_{22}, we write:
$I_{22} \leq \int_{|y| \geq \delta^{-1}} |v^2 - v_{null}^2| |y|^2 \rho dy \leq \delta^{k'-2} \int v^2 |y|^{k'} \rho dy + \delta^{k'-2} \int v_{null}^2 |y|^{k'} \rho dy$
$\leq c_0(k') \delta^2 \int v_{null}^2 \rho dy + \delta^{k'-2} C_{k'}(N)^2 \int v_{null}^2 \rho dy$

by lemma C.1 and the equivalence of norms for v_{null}. Collecting all the above estimates, we get
$I_2 \leq (\delta+2C_4(N)\delta^{3/2}+c_0(k')+\delta^{k'-2}C_{k'}(N)^2)\int v_{null}^2\rho dy$ for $s \leq s_2(\delta)$. If $\delta = \delta(\epsilon)$ is small enough, then
$I_2 \leq \epsilon \int v_{null}^2 \rho dy$ for $s \leq s_3(\epsilon)$.

Now, we handle II_1 and II_2 in the same time: we consider $\delta \in (0, \delta_0)$ and write for $m = 0$ or $m = 2$:
$|\int |g(v)||y|^m\rho dy \leq C\int |v|^3|y|^m\rho dy$
$\leq C\int_{|y|\leq\delta^{-1}} |v|^3|y|^m\rho dy + C\int_{|y|\geq\delta^{-1}} |v|^3|y|^m\rho dy$
$\leq C\epsilon'\delta^{-m}\int_{|y|\leq\delta^{-1}} v^2\rho dy + CM\delta^{k'-m}\int_{|y|\geq\delta^{-1}} v^2|y|^{k'}\rho dy$
$\leq (C\epsilon'\delta^{-m} + CMc_0(k')\delta^{4-m})2\int v_{null}^2\rho dy$
where we used the fact that $v \to 0$ as $s \to -\infty$ in $L^\infty(B(0,\delta^{-1}))$, $|v(y,s)| \leq M$, lemma C.1 and $\int v^2\rho dy \leq \int v_{null}^2\rho dy$.
Now, we can choose $\delta = \delta(\epsilon)$ and then $\epsilon' = \epsilon'(\epsilon)$ such that for $s \leq s_5(\epsilon)$
$\int |g(v)||y|^m\rho dy \leq \epsilon\int v_{null}^2\rho dy$.

Setting $s_0(\epsilon) = \min(s_1(\epsilon), s_3(\epsilon), s_5(\epsilon))$, we have: $\forall\epsilon > 0$, $\forall s \leq s_0(\epsilon)$,
$I_1 + I_2 + II_1 + II_2 \leq 4\epsilon\int v_{null}^2\rho dy$. Therefore $(I) + (II) = o(\|v_{null}(s)\|_{L^2_\rho})$ as $s \to -\infty$.

Thus, combining this with (79) and (85) concludes the proof of Proposition C.1. ■

Step 2: $\|v(s)\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$ as $s \to -\infty$

In this step, we show that although we can not derive directly from (80) the asymptotic behavior of $v_{null}(s)$ (and then the one of $v(s)$), we can use it to show that $\|v(s)\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$ as $s \to -\infty$. More precisely, we have the following Proposition:

Proposition C.2 *If $\|v_-(s)\|_{L^2_\rho} + \|v_+(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$, then for $-s$ large enough, we have*

$$\frac{c}{|s|} \leq \|v(s)\|_{L^2_\rho} \leq \frac{C}{|s|}$$

for some positive constants c and C.

Proof. Since $\|v(s)\|_{L^2_\rho} \sim \|v_{null}(s)\|_{L^2_\rho}$, and because of (82), it is enough to show that

$$\frac{c}{|s|} \leq \|A(s)\| \leq \frac{C}{|s|} \qquad (88)$$

for $-s$ large. The proof is completely parallel to section 3 of Filippas and Liu [4]. Therefore, we give only its main steps.

We first give a result from the perturbation theory of linear operators which asserts that $A(s)$ has continuously differentiable eigenvalues:

Lemma C.2 *Suppose that $A(s)$ is a $N \times N$ symmetric and continuously differentiable matrix-function in some interval I. Then, there exist continuously differentiable functions $\lambda_1(s), ..., \lambda_N(s)$ in I, such that for all $j \in \{1, .., N\}$,*

$$A(s)\phi^{(j)}(s) = \lambda_j(s)\phi^{(j)}(s),$$

for some (properly chosen) orthonormal system of vector-functions $\phi^{(1)}(s), ..., \phi^{(N)}(s)$.

The proof of this lemma is contained (for instance) in Kato [10] or Rellich [13].

We consider then $\lambda_1(s), ..., \lambda_N(s)$ the eigenvalues of $A(s)$. It is well-known that $\sum_{i=1}^{N} |\lambda_i|$ is a norm on the space of $N \times N$ symmetric matrices. We choose this norm to prove (88). From (83), we can derive an equation satisfied by $(\lambda_i(s))_i$:

Lemma C.3 *The eigenvalues of $A(s)$ satisfy $\forall i \in \{1, ..., N\}$*

$$\lambda_i'(s) = \frac{4p}{\kappa}\lambda_i^2(s) + o\left(\sum_{i=1}^{N} \lambda_i^2(s)\right).$$

The proof of lemma 3.3 in [4] holds here with the slight change: $s \to +\infty$ becomes $s \to -\infty$ and s large enough becomes $-s$ large enough.

Now, we claim that with the introduction of $\Lambda_i(\sigma) = -\lambda_i(-\sigma)$, we have:

- $\forall i \in \{1, ..., N\}$

$$\Lambda_i'(\sigma) = \frac{4p}{\kappa}\Lambda_i^2(\sigma) + o\left(\sum_{i=1}^{N} \Lambda_i^2(\sigma)\right) \text{ as } \sigma \to +\infty,$$

-$\forall \sigma \geq \sigma_0$, $\sum_i |\Lambda_i(\sigma)| \neq 0$ (Indeed, if not, then for all i, $\Lambda_i \equiv 0$, $\lambda_i \equiv 0$, and then $A(s)$, $v_{null}(s)$ and $v(s)$ are identically zero.)

Section 3 of [4] yields (directly and without any adaptations) that for all $\sigma \geq \sigma_1$,

$$\frac{c}{\sigma} \leq \sum_i |\Lambda_i(\sigma)| \leq \frac{C}{\sigma}.$$

Since $\|A(s)\| = \sum_i |\lambda_i(s)| = \sum_i |\Lambda_i(-s)|$, this concludes the proof of (88) and the proof of Proposition C.2. ■

Step 3: A new ODE satisfied by $v_{null}(y, s)$

In this step, we show that since $\|v\|_{L^2_\rho}$ behaves like $\frac{1}{|s|}$, then all the L^q_ρ norms are in some sense equivalent as $s \to -\infty$ for this particular v. Then, we will do a kind of center-manifold theory for this particular v to show that $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho}$ is in fact $O(\|v_{null}(s)\|^2_{L^2_\rho})$ and not only $o(\|v_{null}(s)\|_{L^2_\rho})$. These two estimates are used then to rederive a more accurate equation satisfied by $v_{null}(y, s)$.

Lemma C.4 *If $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho} = o(\|v_{null}(s)\|_{L^2_\rho})$, then*

i) for every $r > 1$, $q > 1$, there exists $C = C(r, q)$ such that

$$\left(\int v^r(y,s)\rho dy\right)^{1/r} \leq C\left(\int v^q(y,s)\rho dy\right)^{1/q}$$

for $-s$ large enough.

ii) $\|v_+(s)\|_{L^2_\rho} + \|v_-(s)\|_{L^2_\rho} = O(\|v_{null}(s)\|^2_{L^2_\rho})$ as $s \to -\infty$.

Proof of $i)$ of lemma C.4: The crucial estimate is an a priori estimate of solutions of (38) shown by Herrero and Velazquez in [9]. This a priori estimate is a version of $i)$ holding for all bounded (in L^∞) solutions of (38), but with a delay time; although they proved their result in the case $N = 1$ for solutions defined for $s \in [0, +\infty)$, their proof holds in higher dimensions with $s \in \mathbb{R}$.

Lemma C.5 (Herrero-Velazquez) *Assume that v solves (38) and $|v| \le M < \infty$. Then for any $r > 1$, $q > 1$ and $L > 0$, there exist $s_0^* = s_0^*(q,r)$ and $C = C(r,q,L) > 0$ such that*

$$\left(\int v^r(y, s+s^*)\rho dy\right)^{1/r} \le C\left(\int v^q(y,s)\rho dy\right)^{1/q}$$

for any $s \in \mathbb{R}$ and any $s^* \in [s_0^*, s_0^* + L]$.

Set $s_1^* = s_0^*(2,r)$ and $s_2^* = s_0^*(q,2)$. For $-s$ large enough, we write according to lemma C.5 and Proposition C.2:
$\left(\int v^r(y,s)\rho dy\right)^{1/r} \le C_1\left(\int v^2(y, s-s_1^*)\rho dy\right)^{1/2} \le C_2/(s-s_1^*)$
$\le C_3/(s+s_2^*) \le C_4\left(\int v^2(y,s+s_2^*)\rho dy\right)^{1/2} \le C_5\left(\int v^q(y,s)\rho dy\right)^{1/q}$. Thus, $i)$ of lemma C.4 follows .

Proof of $ii)$ of lemma C.4: We argue as in Step 2 of Appendix A, and use the same notations: $x(s) = \|v_{null}(s)\|_{L^2_\rho}$, $y(s) = \|v_-(s)\|_{L^2_\rho}$, $z(s) = \|v_+(s)\|_{L^2_\rho}$ and $N(s) = \|V^2\|_{L^2_\rho}$. We have already derived (in the proofs of lemmas A.3 and B.4) two differential inequalities satisfied by x and z. By the same techniques (see also [3]), we can show that

$$\begin{aligned} z' &\ge \frac{1}{2}z - CN \\ |x'| &\le CN \\ y' &\le -\frac{1}{2}y + CN. \end{aligned}$$

By $i)$ of lemma C.4, we have $N(s) \le C\|v(s)\|^2_{L^2_\rho} = C(x^2(s) + y^2(s) + z^2(s))$ for large $-s$.

Since $x, y, z \to 0$ as $s \to -\infty$, we can write for $-s$ large:

$$\begin{aligned} z' &\ge \frac{1}{3}z - C(x+y)^2 \\ |x'| &\le C(x+y+z)^2 \\ y' &\le -\frac{1}{3}y + C(x+z)^2. \end{aligned}$$

The conclusion then follows form the following ODE lemma by Filippas and Liu:

Lemma C.6 (Filippas-Liu) *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions which are non negative and satisfy*

i) $(x,y,z)(s) \to 0$ as $s \to -\infty$,

ii) $\forall s \le s_0$

$$\begin{cases} \dot z \ge c_0 z - c_1(x+y)^2 \\ |\dot x| \le c_1(x+y+z)^2 \\ \dot y \le -c_0 y + c_1(x+z)^2. \end{cases}$$

for some positive constants c_0 *and* c_1. *Then,*
either (i) $x, y, z \to 0$ *exponentially fast as* $s \to -\infty$,
or (ii) for all $s \le s_1$, $y + z \le b(c_0, c_1)x^2$ *for some* $s_1 \le s_0$.

Proof: see lemma 4.1 in [4].

Now, using lemma C.4, we derive a new equation satisfied by v_{null}:

Proposition C.3 $\forall i, j \in \{1, ..., N\}$,

$$v'_{2,ij}(s) = \frac{p}{2\kappa} \int v_{null}^2(y,s) \frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}} \rho(y)dy + O(\|v_{null}(s)\|^3_{L^2_\rho}).$$

as $s \to -\infty$.

Moreover,

$$A'(s) = \frac{4p}{\kappa} A^2(s) + O(\frac{1}{s^3}) \text{ as } s \to -\infty.$$

The proof of Proposition 4.1 in [4] holds here with the usual changes: $s \to +\infty$ becomes $s \to -\infty$.

Step 4: Asymptotic behavior of $v_{null}(y,s)$, $v_0(s)$ and $v_1(s)$
Setting $\mathcal{A}(\sigma) = -A(-\sigma)$, we see that

$$\mathcal{A}'(\sigma) = \frac{4p}{\kappa}\mathcal{A}^2(\sigma) + O(\frac{1}{\sigma^3}) \text{ as } \sigma \to +\infty.$$

Therefore, Proposition 5.1 in [4] yields (directly and without any adaptations) the existence of $\delta > 0$ and a $N \times N$ orthonormal matrix Q such that

$$\mathcal{A}(\sigma) = -\frac{\kappa}{4p\sigma}A_0 + O(\frac{1}{\sigma^{1+\delta}})$$

where

$$A_0 = Q \begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$$

for some $k \in \{0, 1, ..., N-1\}$. Together with (81), this yields the behavior of $v_{null}(y,s)$ announced in $i)$ of Proposition 3.7.

It also yields

$$\begin{aligned}
\|v_{null}(s)\|_{L^2_\rho} &= \left(\int v_{null}^2(y,s)\rho(y)dy\right)^{1/2} \\
&= \left(\int (y^T A(s) y - 2trA(s))^2 \rho(y)dy\right)^{1/2} \\
&= -\frac{\kappa}{4ps}\left(\int (y^T A_0 y - 2trA_0)^2\rho(y)dy\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right).
\end{aligned}$$

With the change of variables, $y = Qz$, we get since Q is orthonormal:

$$\|v_{null}(s)\|_{L^2_\rho} = -\frac{\kappa}{4ps}\left(\int \left(\sum_{i=1}^{N-k}(z_i^2 - 2)\right)^2 \rho(z)dz\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right)$$

$$= -\frac{\kappa}{4ps}\left(\sum_{i=1}^{N-k}\int (z_i^2-2)^2\rho(z)dz\right)^{1/2} + O\left(\frac{1}{|s|^{1+\delta}}\right)$$
$$= -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right)$$

where we used the fact that $(y_i^2-2)_i$ is an orthogonal system with respect to the measure ρdy.

Since $\|v(s)\|_{L^2_\rho} = \|v_{null}(s)\|_{L^2_\rho} + O\left(\|v_{null}(s)\|^2_{L^2_\rho}\right)$ (ii) of lemma C.4), we get

(89) $$\|v(s)\|_{L^2_\rho} = -\frac{\kappa}{ps}\sqrt{\frac{N-k}{2}} + O\left(\frac{1}{|s|^{1+\delta}}\right).$$

Integrating (38) with respect to ρdy, we find

$$v_0'(s) = v_0(s) + \int f(v)\rho dy.$$

Since $|f(v)| \le Cv^2$, we get from (89)

$$v_0'(s) = v_0(s) + O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

Therefore, it follows that

$$v_0(s) = O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

Using lemma C.1, we have: for all $\eta \in (0,\delta_0)$,

$$\int v^2|y|^{k'}\rho dy \le c_0(k')\eta^{4-k'}\int v_{null}^2\rho dy \le 2c_0(k')\eta^{4-k'}\int v^2\rho dy.$$

Therefore,
$\int v^2|y|\rho dy \le \int_{|y|\le\eta^{-1}} v^2|y|\rho dy + \int_{|y|\ge\eta^{-1}} v^2|y|\rho dy$
$\le \eta^{-1}\int v^2\rho dy + \eta^{k'-1}\int v^2|y|^{k'}\rho dy$
$\le (\eta^{-1}+2c_0(k')\eta^3)\int v^2\rho dy.$

If we fix $\eta > 0$, then

(90) $$\int v^2|y|\rho dy \le C(\eta,k')\int v^2\rho dy.$$

Integrating (38) with respect to $y_i\rho dy$, we find

$$v_{1,i}'(s) = \frac{1}{2}v_{1,i}(s) + \int f(v)\frac{y_i}{2}\rho dy.$$

Since $|f(v)| \le Cv^2$, we get from (90) and (89)

$$v_1(s) = O(\frac{1}{s^2}) \text{ as } s \to -\infty.$$

This concludes the proof of i) of Proposition 3.7.

Acknowledgment

The second author is thankful for the invitation of the Institute for Advanced Study where part of this work was done.

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Chapitre 3

Refined uniform estimates at blow-up and applications for nonlinear heat equations

Refined uniform estimates at blow-up and applications for nonlinear heat equations†

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1 Introduction

We are interested in the following nonlinear heat equation:

$$\begin{cases} u_t & = & \Delta u + u^p \\ u(0) & = & u_0 \geq 0, \end{cases} \tag{1}$$

where u is defined for $(x,t) \in \mathbb{R}^N \times [0,T)$, $1 < p$, $(N-2)p < N+2$ and $u_0 \in H^1(\mathbb{R}^N)$.

In this paper, we deal with blow-up solutions of equation (1) $u(t)$ which blow-up in finite time $T > 0$: this means that u exists for all $t \in [0,T)$, $\lim\limits_{t\to T} \|u(t)\|_{H^1} = +\infty$ and $\lim\limits_{t\to T} \|u(t)\|_{L^\infty} = +\infty$. Let us consider such a solution. We aim at studying the blow-up behavior of $u(t)$ as $t \to T$. In particular, we are interested in obtaining uniform estimates on $u(t)$ and deducing from these estimates the asymptotic shape of the singularities.

One can show that in this case, $u(t)$ has at least one blow-up point, that is $x_0 \in \mathbb{R}^N$ such that there exists $(x_n, t_n)_{n\in\mathbb{N}}$ satisfying $(x_n, t_n) \to (x_0, T)$ and $|u(x_n, t_n)| \to +\infty$ as $n \to +\infty$.

For each $a \in \mathbb{R}^N$, we introduce the following self-similar transformation:

$$\begin{array}{rcl} y & = & \frac{x-a}{\sqrt{T-t}} \\ s & = & -\log(T-t) \\ w_a(y,s) & = & (T-t)^{\frac{1}{p-1}} u(x,t). \end{array} \tag{2}$$

Then, we see that $w_a = w$ satisfies $\forall s \geq -\log T$, $\forall y \in \mathbb{R}^N$:

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + w^p. \tag{3}$$

The study of $u(t)$ near (x_0, T) where x_0 is a blow-up point is equivalent to the study of the long time behavior of w_{x_0} as $s \to +\infty$.
Giga and Kohn prove in [10] that there exists $\epsilon_0 > 0$ such that

$$\forall s \geq -\log T,\ \epsilon_0 \leq \|w_{x_0}(s)\|_{L^\infty} \leq \frac{1}{\epsilon_0}$$

or equivalently:

$$\forall t \in [0,T),\ \epsilon_0 (T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq \frac{1}{\epsilon_0} (T-t)^{-\frac{1}{p-1}}.$$

† article à paraître dans Geom. Funct. Anal.

At this level, no other uniform estimates were known.

In [16], we proved the following Liouville Theorem for equation (3):

Let w be a nonnegative solution of (3) defined for $(y,s)\in\mathbb{R}^N\times\mathbb{R}$ such that $w\in L^\infty(\mathbb{R}^N\times\mathbb{R})$. Then, necessarily one of the following cases occurs:

$$w\equiv 0 \text{ or } w\equiv\kappa \text{ or } \exists s_0\in\mathbb{R} \text{ such that } w(y,s)=\varphi(s-s_0) \tag{4}$$

$$\text{where } \varphi(s)=\kappa(1+e^s)^{-\frac{1}{p-1}} \text{ and } \kappa=(p-1)^{-\frac{1}{p-1}}.$$

From this theorem we derived in [16] the following uniform estimates of order zero:

Consider a solution w of (3) defined for $s\geq-\log T$ (such that $u(t)$ blows-up at time T). Then,

$$\|w(s)\|_{L^\infty}\to\kappa \text{ and } \|\nabla w(s)\|_{L^\infty}+\|\Delta w(s)\|_{L^\infty}\to 0 \text{ as } s\to+\infty. \tag{5}$$

We also derived from this result the following localization theorem:

$\forall\epsilon>0$, $\exists C_\epsilon>0$ *such that* $\forall t\in[\frac{T}{2},T)$, $\forall x\in\mathbb{R}^N$,

$$\left|\frac{\partial u}{\partial t}-u^p\right|\leq\epsilon u^p+C_\epsilon. \tag{6}$$

These estimates are still insufficient to yield precise estimates on blow-up profile. But, we have a compactness property on $w_a(s)$ uniformly with respect to $a\in\mathbb{R}^N$, which allows us to claim the following result from linearization around the limit set as $s\to+\infty$:

Theorem 1 (Refined L^∞ estimates for $w(s)$ and $u(t)$ at blow-up) *There exist positive constants C_1, C_2 and C_3 such that if u is a solution of (1) which blows-up at time $T>0$ and satisfies $u(0)\in H^1(\mathbb{R}^N)$, then $\forall\epsilon>0$, there exists $s_0(\epsilon)\geq-\log T$ such that*

i) $\forall s\geq s_0$, $\forall a\in\mathbb{R}^N$,

$$\begin{array}{rclrcl}\|w_a(s)\|_{L^\infty} & \leq & \kappa+(\frac{N\kappa}{2p}+\epsilon)\frac{1}{s}, & \|\nabla w_a(s)\|_{L^\infty} & \leq & \frac{C_1}{\sqrt{s}},\\ \|\nabla^2 w_a(s)\|_{L^\infty} & \leq & \frac{C_2}{s}, & \|\nabla^3 w_a(s)\|_{L^\infty} & \leq & \frac{C_3}{s^{3/2}},\end{array}$$

where $\kappa=(p-1)^{-\frac{1}{p-1}}$,

ii) $\forall t\geq T-e^{-s_0}$,

$$\begin{array}{rcl}\|u(t)\|_{L^\infty} & \leq & \left(\kappa+(\frac{N\kappa}{2p}+\epsilon)\frac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}},\\ \|\nabla^i u(t)\|_{L^\infty} & \leq & C_i\frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{i/2}}\end{array}$$

for $i=1,2,3$.

Remark: If $v:\mathbb{R}^N\to\mathbb{R}$ is regular, $\nabla^i v$ stands for the differential of order i of v. For all $y\in\mathbb{R}^N$, we define $|\nabla v(y)|^2=\sum_{j=1}^N(\partial_j v(y))^2$, $|\nabla^2 v(y)|=$

$\sup_{z\in\mathbb{R}^N}\frac{|z^T\nabla^2 v(y)z|}{|z|^2}$ and $|\nabla^3 v(y)| = \sup_{\alpha,\beta,\gamma\in\mathbb{R}^N}\left|\sum_{i,j,k}\frac{\alpha_i}{|\alpha|}\frac{\beta_j}{|\beta|}\frac{\gamma_k}{|\gamma|}\partial^3_{i,j,k}v(y)\right|$.
In addition, $\|v\|_{L^\infty} = \sup_{y\in\mathbb{R}^N}|v(y)|$ and $\|\nabla^i v\|_{L^\infty} = \sup_{y\in\mathbb{R}^N}|\nabla^i v(y)|$.

In fact, we can see from the proof of Theorem 1 that $s_0(\epsilon)$ depends only on the size of initial data. We have the following result:

Theorem 1' (Compactness) *Consider $(u_n)_{n\in\mathbb{N}}$ a sequence of nonnegative solutions of equation (1) such that for some $T>0$ and for all $n\in\mathbb{N}$, u_n is defined on $[0,T)$ and blows-up at time T. Assume also that $\|u_n(0)\|_{H^2(\mathbb{R}^N)}$ is bounded uniformly in n. Then, $\forall\epsilon>0$, there exists $t_0(\epsilon)<T$ such that $\forall t\in[t_0(\epsilon),T)$, $\forall n\in\mathbb{N}$,*

$$\begin{aligned}\|u_n(t)\|_{L^\infty} &\leq \left(\kappa + \left(\frac{N\kappa}{2p}+\epsilon\right)\frac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}},\\ \|\nabla^i u_n(t)\|_{L^\infty} &\leq C_i\frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{\frac{i}{2}}}\end{aligned}$$

where C_i are defined in Theorem 1.

Remark: In the case $N=1$, Herrero and Velázquez [12] (Filippas and Kohn [6] also) prove some estimates related to Theorem 1, using a Sturm property first used by Chen and Matano [4] (the space oscillations number is a decreasing function of time).
Remark: The constant $\frac{N\kappa}{2p}$ appearing in the term of order one in the estimates on $\|w(s)\|_{L^\infty}$ and $\|u(t)\|_{L^\infty}$ is optimal. Indeed, there exist solutions of equation (3) such that $\|w(s)\|_{L^\infty} = \kappa + \frac{N\kappa}{2ps} + o\left(\frac{1}{s}\right)$ as $s\to+\infty$ (see Bricmont and Kupiainen [3], Filippas and Kohn [6], Merle and Zaag [17]).
Remark: From the local (in time) regularity of the solution to the Cauchy problem, we can obtain with the same proof an analogous compactness result when the blow-up times are not the same. One has to replace T by T_n and $[\exists t_0(\epsilon)$ such that $\forall t\in[t_0(\epsilon),T)]$ by $[\exists t_0'(\epsilon)$ such that $\forall n\in\mathbb{N}$, $\forall t\in[T_n-t_0'(\epsilon),T_n)]$.
Remark: Other compactness results can be shown considering for example equations of the type:
$$\frac{\partial u}{\partial t} = \Delta u + b(x)u^p$$
where $b\in C^3(\mathbb{R}^N)$ (see [16]).

These estimates are in fact crucial for the understanding of the solution at blow-up, especially, the shape of the singularity. Let us recall some results on this question.

Let us consider $x_0\in\mathbb{R}^N$ a blow-up point of $u(t)$, a solution of (1), that is a point $x_0\in\mathbb{R}^N$ such that there exists $(x_n,t_n)\to(x_0,T)$ such that $u(x_n,t_n)\to+\infty$ as $n\to+\infty$. The question is to see whether $u(t)$ (or $w_{x_0}(s)$ defined in (2)) has a universal behavior as $t\to T$ (or $s\to+\infty$).

First, Giga and Kohn prove in [10] and [11] (see also [9]) that for a given blow-up point $x_0\in\mathbb{R}^N$,

$$\lim_{s\to+\infty} w_{x_0}(y,s) = \lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(x_0+y\sqrt{T-t},t) = \kappa \tag{7}$$

uniformly on compact subsets of $\mathbb{R}^N$. The result is pointwise in x_0. Besides, for a.e. y, $\lim_{s\to+\infty} \nabla w_{x_0}(y,s) = 0$.
Filippas and Liu [7] (see also Filippas and Kohn [6]) and Velázquez [18], [19] (see also Herrero and Velázquez [12], [14]) classify the behavior of $w(y,s)$ ($= w_{x_0}(y,s)$) for $|y|$ bounded. They prove that one of the following cases occurs:

- Case 1: non degenerate rate of blow-up:

there exists $k \in \{0,1,...,N-1\}$ and a $N \times N$ orthonormal matrix Q such that $\forall R > 0$,

$$\sup_{|y|\le R} \left| w_{x_0}(y,s) - \left[\kappa + \frac{\kappa}{2ps}\left((N-k) - \frac{1}{2} y^T A_k y\right)\right]\right| = O\left(\frac{1}{s^{1+\delta}}\right) \tag{8}$$

as $s \to +\infty$ where $\delta > 0$,

$$A_k = Q \begin{pmatrix} I_{N-k} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} \tag{9}$$

and I_{N-k} is the $(N-k)\times(N-k)$ identity matrix,

- Case 2: degenerate rate of blow-up: $\forall R > 0$, $\sup_{|y|\le R} |w(y,s) - \kappa| \le C(R) e^{-\epsilon_0 s}$

for some $\epsilon_0 > 0$.
This yields a blow-up behavior classification in a small range scale. In some sense and from a physical point of view, these results do not show the transition between the singular zone ($w \ge \alpha$ where $\alpha > 0$) and the regular one ($w \simeq 0$) well.
Using the renormalization theory, Bricmont and Kupiainen showed in [3] the existence of a solution of (3) such that

$$\forall s \ge s_0,\ \forall y \in \mathbb{R}^N,\ \left| w(y,s) - f_0\left(\frac{y}{\sqrt{s}}\right)\right| \le \frac{C}{\sqrt{s}} \tag{10}$$

where $f_0(z) = \left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}$ (see also [1]). We show in [17] the same result through a reduction to a finite dimensional problem. We also obtained there a stability result of this behavior with respect to initial data. This gives a result in an intermediate scale $z = \frac{y}{\sqrt{s}}$, which is more satisfactory since it separates the blow-up region ($w > \alpha > 0$) and non-blow-up ones ($w \simeq 0$).
In [20], the second author showed that the behavior in the initial variable x is known in the case where (10) occurs. More precisely, $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$ and

$$u^*(x) \sim \left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}} \quad \text{as } x \to 0. \tag{11}$$

Therefore, except in the small range variable (which does not precise from a physical or analytical point of view the singular behavior), no result of classification was known.

In a first step, we use the estimates of Theorem 1 on ∇w and $\nabla^2 w$ in a crucial way, and the results of Filippas and Liu, and Velázquez concerning the classification of blow-up behaviors for $|y|$ bounded to establish a blow-up profile classification theorem in the variable $z = \frac{y}{\sqrt{s}}$ (which is the intermediate scale that separates the regular and singular parts in the non degenerate case):

Theorem 2 (Existence of a blow-up profile in the intermediate scale for solutions of (1))
Let $u(t)$ be a solution of (1) which blows-up at time $T > 0$ and satisfies $u(0) \in H^1(\mathbb{R}^N)$. Let x_0 be a blow-up point of $u(t)$. Then, there exist $k \in \{0, 1, .., N\}$ and an orthonormal $N \times N$ matrix Q such that

$$\forall K_0 > 0, \quad \sup_{|z| \le K_0} |w_{x_0}(z\sqrt{s}, s) - f_k(z)| \to 0 \text{ as } s \to +\infty, \tag{12}$$

where

$$f_k(z) = \left(p - 1 + \frac{(p-1)^2}{4p} z^T A_k z\right)^{-\frac{1}{p-1}} \tag{13}$$

and A_k is defined in (9).

Remark: Velázquez in [19] obtained a related profile existence result. He extended the $|y|$ bounded convergence of [18] to the larger set $|y| \le K_0\sqrt{s}$, by estimating the effect of the convective term $-\frac{1}{2}y.\nabla w$ in the equation (3), in L^p spaces with a Gaussian measure. However, the convergence that he obtains depends strongly on the considered blow-up point x_0. Let us point out that the convergence we have in Theorem 2 can be shown to be independent of x_0. Indeed, by using the uniform estimates of Theorem 1, we can give a uniform version of the result of [7] and [18], and obtain thanks to our techniques a convergence independent of x_0 in Theorem 2. However, we use the result of [7] and [18] in this paper, since this shortens the proof. We also notice that the proof yields that if the case (12) occurs, then (8) occurs with the same A_k (if $k = N$, then take $A_N = 0$) and conversely. See also Theorem 3.
Remark: In the case $k = N$, this theorem yields κ as asymptotic "profile" of $w(s)$ in the variable $z = \frac{y}{\sqrt{s}}$: this is a degenerate blow-up behavior. Indeed, in this case, the scale $\frac{y}{\sqrt{s}}$ is not good for describing the blow-up behavior. One must refine this scale and exhibit other blow-up profiles in different scales $y \simeq \exp\left[\left(\frac{k-1}{2k}\right) s\right]$ for $k = 2, 3, ...$ (see for instance [3], [18]). However, we suspect these profiles to be unstable with respect to initial data.

One interesting problem that follows from Theorem 2 is to find a relationship between the different notions of profile in the scales: $|y| \le C$, $z = \frac{|y|}{\sqrt{s}} \le C$ and $|x - x_0|$ small. We show in the following theorem that all these descriptions are equivalent in the case of a solution $u(t)$ of (1) that blows-up at some point $x_0 \in \mathbb{R}^N$ in a non degenerate way (which is supposed to be the generic case):

$$k = 0 \text{ and } A_k = I_N.$$

This answers many questions which were underlined on this problem in preceding works.

Theorem 3 (Equivalence of different notions of blow-up profiles)
Let $x_0 \in \mathbb{R}^N$ be an isolated blow-up point of $u(t)$ solution of (1) such that $u_0 \in H^1(\mathbb{R}^N)$. The following blow-up behaviors of $u(t)$ near x_0 or $w(s) = w_{x_0}(s)$ (defined in (2)) are equivalent:

(A) $\forall R > 0,\ \sup_{|y|\le R} \left| w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ *as* $s \to +\infty$ *where* $\kappa = (p-1)^{-\frac{1}{p-1}}$,

(B) $\exists \epsilon_0 > 0$ *such that* $\left\| w(y,s) - f_0(\frac{y}{\sqrt{s}})\right\|_{L^\infty(|y|\le \epsilon_0 e^{s/2})} \to 0$ *as* $s \to +\infty$ *with* $f_0(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

(C) $\exists \epsilon_0 > 0$ *such that if* $|x - x_0| < \epsilon_0$, *then* $u(x,t) \to u^*(x)$ *as* $t \to T$ *and* $u^*(x) \sim \left[\frac{8p|\log|x-x_0||}{(p-1)^2|x-x_0|^2}\right]^{\frac{1}{p-1}}$ *as* $x \to x_0$.

Remark: In [19], Velázquez shows that (A) $\Longrightarrow$ (B) $\Longrightarrow$ (C) by estimating the local effect to the term $-\frac{1}{2}y.\nabla w$ in equation (3) in L^p with Gaussian measure. The classification of [19] also yields that (C) $\Longrightarrow$ (A). Let us point that the estimates in our proof are quite elementary and rely on localization effect and uniform estimates. In addition, one can show from our proof and our uniform techniques that the convergence speeds in (A), (B) and (C) depend only on each other and on a bound on the C^2 norm of initial data (and not on the initial data itself).

Remark: In fact, (A) (or (B) or (C)) imply that x_0 is an isolated blow-up point. It is conjectured that the equivalence holds (in the case of the (supposed to be) generic blow-up rate).

Remark: The techniques we introduce in the proof of Theorem 3 allow us to obtain the same results as Velázquez in the case where (8) occurs with $k < N$.

Section 2 is devoted to the proof of the uniform estimates on w (Theorems 1 and 1'). Section 3 deals with results on profiles (Theorems 2 and 3).

2 L^∞ estimates of order one for solutions of (3)

2.1 Formulation and reduction of the problem

We prove Theorems 1 and 1' in this section. Let us first show Theorem 1. Theorem 1' follows from similar arguments.

Proof of Theorem 1: We consider $u(t)$ a blow-up solution of (1) which blows-up at time $T > 0$.

We can assume from regularizing effect of the heat flow that $T < 1$, $u_0 \in C^3(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. We are interested in finding L^∞ estimates of order one for w_0 $(= w)$ defined in (2). In [16], we have already proved L^∞ estimates of order zero for w stated in (5). Note that with obvious simple adaptations of the proof of (5), we have the following result:

(14) $\|w(s)\|_{L^\infty} \to \kappa$ and $\|\nabla w(s)\|_{L^\infty} + \|\nabla^2 w(s)\|_{L^\infty} + \|\nabla^3 w(s)\|_{L^\infty} \to 0$

as $s \to +\infty$.

We now want to refine the estimates (14). More precisely, we want to show that there exist positive constants C_1, C_2 and C_3 depending only on p such that $\forall \epsilon > 0$, $\exists s_0(\epsilon)$ such that $\forall s \ge s_0(\epsilon)$,

$$(15)\quad \begin{array}{rclrcl} \|w(s)\|_{L^\infty} & \le & \kappa + (\frac{N\kappa}{2p} + (N+1)\epsilon)\frac{1}{s}, & \|\nabla w(s)\|_{L^\infty} & \le & \frac{C_1}{\sqrt{s}} \\ \|\nabla^2 w(s)\|_{L^\infty} & \le & \frac{C_2}{s}, & \|\nabla^3 w(s)\|_{L^\infty} & \le & \frac{C_3}{s^{3/2}}. \end{array}$$

For this purpose, we take an arbitrary $\epsilon \in (0,\epsilon_0)$ (where $\epsilon_0 \le 1$ is small enough) that we consider as fixed now, and introduce the following definitions:

Definition 2.1 *For all $A > 0$ and $s \ge -\log T$, we define $V_A(s)$ as being the set of all $w \in W^{3,\infty}(\mathbb{R}^N)$ satisfying:*

$$\begin{array}{rclrcl} \|w\|_{L^\infty} & \le & \kappa + \frac{c_0}{s}, & \|\nabla w\|_{L^\infty} & \le & \frac{c_1}{\sqrt{s}} \\ \|\nabla^2 w\|_{L^\infty} & \le & \frac{A}{s}, & \|\nabla^3 w\|_{L^\infty} & \le & \frac{A^{5/4}}{s^{3/2}}. \end{array}$$

and

$$\forall a \in \mathbb{R}^N, \quad -\frac{c_2}{s} I_N \le \int_{\mathbb{R}^N} \nabla^2 w(y+a)\rho(y)dy$$

in the sense of symmetric $N \times N$ matrices, where the norms are introduced in the remark after Theorem 1,

$$c_0(\epsilon) = \frac{N\kappa}{2p} + (N+1)\epsilon, \; c_1(\epsilon) = \frac{\kappa}{\sqrt{p}} + 2\epsilon\sqrt{p}, \; c_2(\epsilon) = \frac{\kappa}{2p} + \epsilon, \tag{16}$$

$$I_N \textit{ is the } N \times N \textit{ identity matrix and } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \tag{17}$$

Definition 2.2 *For all $s \ge -\log T$, we define*
$\hat{V}_A(s) = \{w \in C([-\log T, s), W^{3,\infty}(\mathbb{R}^N)) \mid \forall \tau \in [-\log T, s),\ w(\tau) \in V_A(\tau)\}$.

Let us remark that condition (16) is in some sense a lower bound on $\nabla^2 w(a)$. Indeed, if $w \in V_A(s)$, then we have $\forall a \in \mathbb{R}^N$,

$$\left|\int_{\mathbb{R}^N} \nabla^2 w(y+a)\rho(y)dy - \nabla^2 w(a)\right| \le C^*(N)\|\nabla^3 w\|_{L^\infty} \tag{18}$$

$$\text{and} \qquad \frac{A}{s} I_N \ge \nabla^2 w(a) \ge -\left[\frac{c_2}{s} + C^*(N)\frac{A^{5/4}}{s^{3/2}}\right] I_N \tag{19}$$

where $C^*(N) = \int |y|\rho(y)dy$.
Proof of (18) and (19): Using a Taylor expansion, we have: $\forall y \in \mathbb{R}^N$, $\nabla^2 w(y+a) - \nabla^2 w(a) = \int_0^1 \nabla^3 w(a+ty)(y)dt$. Hence,
$|\nabla^2 w(y+a) - \nabla^2 w(a)| \le |y|\|\nabla^3 w\|_{L^\infty} \le |y|\frac{A^{5/4}}{s^{3/2}}$. This yields (18) and (19) by integration (use $\int \rho(y)dy = 1$). ■

Notice that the lower bound on $\nabla^2 w(a)$ is (consider the order $\frac{1}{s}$) independent of A, which will be crucial in the proof.

Theorem 1 is in fact a consequence of the following proposition:

Proposition 2.1 (Reduction) *There exist $A(p) > 0$ and $\epsilon_0(p) \in (0,1)$ such that for all $\epsilon \in (0,\epsilon_0)$, there exists $S(A,\epsilon)$ so that the following property is true: Assume that w is a solution of (3) defined for all time $s \ge -\log T$ and satisfying $w(-\log T) \in H^1(\mathbb{R}^N)$. Assume in addition that $w \in \hat{V}_A(\hat{s})$ for some $\hat{s} \ge S(A,\epsilon)$, then:*
i) $w(\hat{s}) \notin \partial V_A(\hat{s})$,
ii) $\forall s \ge -\log T$, $w(s) \in \hat{V}_A(s)$.

Proposition 2.1 implies Theorem 1:

Let $\epsilon \in (0, \epsilon_0)$, $A = A(p)$ and $S(A,\epsilon)$ defined in Proposition 2.1. Our strategy is to find $n_0(\epsilon) = n_0 \in \mathbb{N}$ such that $\forall s \geq -\log T$, $w(s+n_0) \in V_A(s)$. Indeed, one can easily check the following result:

Lemma 2.1 *Assume for all $\epsilon \in (0,1)$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that $\forall s \geq -\log T$, $w(s+n_0) \in V_A(s)$. Then, (15) is satisfied with*

$$C_1 = \frac{\kappa}{2p} + 4\sqrt{p},\ C_2 = 2A(p) \text{ and } C_3 = 2A(p)^{5/4}.$$

Let us consider $W = w(.+n)$. Then, W satisfies (3) for all $s \geq -\log T$ and $W(-\log T) = w(n - \log T) \in H^1(\mathbb{R}^N)$ from the solving of the initial value problem for w.

We claim the following: for n large, we have $w(.+n) \in \hat{V}_A(S(A,\epsilon))$. Indeed, let $\delta =$

$$(20)\quad \frac{1}{4(1+C^*(N))}\min\left(\frac{c_0}{S(A,\epsilon)}, \frac{c_1}{\sqrt{S(A,\epsilon)}}, \frac{c_2}{S(A,\epsilon)}, \frac{A}{S(A,\epsilon)}, \frac{A^{5/4}}{S(A,\epsilon)^{3/2}}\right)$$

where $C^*(N)$ is defined in (19). (14) implies that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\forall s \in [-\log T, S(A,\epsilon)]$, $\|w(s+n)\|_{L^\infty} \leq \kappa + \delta \leq \kappa + \frac{c_0}{4s}$, $\|\nabla w(s+n)\|_{L^\infty} \leq \delta \leq \frac{c_1}{4\sqrt{s}}$, $\|\nabla^2 w(s+n)\|_{L^\infty} \leq \delta \leq \frac{A}{4s}$ and $\|\nabla^3 w(s+n)\|_{L^\infty} \leq \delta \leq \frac{A^{5/4}}{4s^{3/2}}$.

Let $s \in [-\log T, S(A,\epsilon)]$ and $a \in \mathbb{R}^N$. According to (18), we have $\int_{\mathbb{R}^N} \nabla^2 w(y+a, s+n)\rho(y)dy \geq -\left(|\nabla^2 w(a, s+n)| + C^*(N)\|\nabla^3 w(s+n)\|_{L^\infty}\right) I_N \geq -(\delta + C^*(N)\delta)\, I_N \geq -\frac{c_2}{4s_2} I_N$. Thus, $w(.+n_0) \in \hat{V}_A(S(A,\epsilon))$. Applying Proposition 2.1, we see from *ii)* that

$$\forall s \in [-\log T, +\infty),\ w(s+n_0) \in V_A(s).$$

This concludes the proof of Theorem 1. ■

Proof of Theorem 1':

For all $n \in \mathbb{N}$, we introduce $w_n = w_{n,0}$ defined from u_n by (2). Then, by simple obvious adaptations of the proof of Theorem 1' in [16], we claim that $\sup_{n\in\mathbb{N}} \|w_n(s)\|_{L^\infty} \to \kappa$ and $\sup_{n\in\mathbb{N}} \|\nabla^i w_n(s)\|_{L^\infty} \to 0$ as $s \to +\infty$ for $i = 1, 2$ and 3.

Hence, there exists $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $\forall s \in [-\log T, S(A,\epsilon)]$, $\|w_n(s+n_0)\|_{L^\infty} \leq \kappa + \delta$ and $\|\nabla^i w_n(s+n_0)\|_{L^\infty} \leq \delta$ for $i = 1,2,3$ where δ is defined in (20). Hence, as for the proof of Theorem 1, we get $\forall n \in \mathbb{N}$, $w_n(.+n_0) \in \hat{V}_A(S(A,\epsilon))$. Thus,

$$\forall n \in \mathbb{N},\ \forall s \in [-\log T, +\infty),\ w_n(s+n_0) \in V_A(s)$$

by *ii)* of Proposition (2.1). This concludes the proof of Theorem 1'. ■

Therefore, the question reduces to prove Proposition 2.1.

Proof of Proposition 2.1:

i) $\Longrightarrow$ *ii)*: By contradiction, we assume that there exists $s \geq -\log T$ such that $w(s) \notin V_A(s)$. Let s' be the lowest s satisfying this. Then, $s' \geq \hat{s} \geq S(A,\epsilon)$, $w \in \hat{V}_A(s')$ and $w(s') \in \partial V_A(s')$. This contradicts *i)*.

Proof of i): Let us argue by contradiction. We suppose that for all $A > 0$, there is a sequence $s_n \to +\infty$ and a solution of (3) w_n defined for all $s \geq -\log T$ such that $w_n(-\log T) \in H^1(\mathbb{R}^N)$, $\forall s \in [-\log T, s_n]$, $w_n(s) \in V_A(s)$ and $w_n(s_n) \in \partial V_A(s_n)$.
Let us denote w_n by w to simplify the notations. We claim the following

Proposition 2.2 (Characterization of $\partial V_A(s_n)$) *There exists $y_n \in \mathbb{R}^N$ such that one of the following cases must occur:*
Case 1: $w(y_n, s_n) = \kappa + \frac{c_0}{s_n}$,
Case 2: $|\nabla w(y_n, s_n)| = \frac{c_1}{\sqrt{s_n}}$,
Case 3: there exists a unitary $\varphi_n \in \mathbb{R}^N$ such that
$\varphi_n^T \int_{\mathbb{R}^N} \nabla^2 w(y + y_n, s_n)\rho(y)dy\varphi_n = -\frac{c_2}{s_n}$,
Case 4: $|\nabla^2 w(y_n, s_n)| = \frac{A}{s_n}$,
Case 5: $|\nabla^3 w(y_n, s_n)| = \frac{A^{5/4}}{s_n^{3/2}}$.

Proof:
Let us remark that since $w(-\log T) \in H^1(\mathbb{R}^N)$, we can assume from the regularizing effect of the heat flow that $w(-\log T, y) \to 0$ and $\nabla^i w(-\log T, y) \to 0$ as $|y| \to +\infty$ for $i = 1, 2$ and 3. Hence, we have by classical estimates $w(y, s) \to 0$ and $\nabla^i w(y, s) \to 0$ as $|y| \to +\infty$ uniformly in $s \in [s_n, s_n + 1]$. Hence, by Lebesgue's Theorem, $\int \nabla^2 w(y + a, s)\rho(y)dy \to 0$ as $|a| \to +\infty$.

This insures that one of the five cases of Proposition 2.2 occurs. ∎

We now use the classification of Proposition 2.2 and consider in the following subsection all the five cases in order to reach a contradiction.

Let us notice that we reduce to the case

$$y_n = 0.$$

Indeed, from (2) and the translation invariance of (1), we define for all $y \in \mathbb{R}^N$ and $s \geq -\log T$:

$$W(y, s) = w(y + y_n e^{\frac{s - s_n}{2}}, s). \tag{21}$$

We still have:
- W is solution of (3) defined for $s \in [-\log T, +\infty)$,
- $W(s) \in V_A(s)$ for all $s \in [-\log T, s_n]$,
- $W(s_n) \in \partial V_A(s_n)$.

We will denote W by w and φ_n by φ.

We now claim that there exist $\epsilon_0(p) > 0$ and $A(p) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, there is $S(A, \epsilon)$ such that all the cases 1, 2, 3, 4 and 5 do not occur if $s_n \geq S(A, \epsilon)$, which will conclude the proof of Proposition 2.1.

2.2 Proof of the boundary estimates

There exist $\epsilon_0(p)$ and $A_0(p)$ such that $\forall \epsilon \in (0, \epsilon_0)$, $\forall A \geq A_0(p)$, $\exists S = S(A, \epsilon)$ such that Cases 1,2,3,4 and 5 do not occur if $s_n \geq S(A, \epsilon)$.

Let us show the following lemma

Lemma 2.2 (Taylor expansions) *Assume that $w(s) \in V_A(s)$. Then, $\forall y \in \mathbb{R}^N$:*

$$(22) - \frac{|y|^2}{2}\left(\frac{c_2}{s} + C^*(N)\frac{A^{5/4}}{s^{3/2}}\right) \le w(y,s) - w(0,s) - y.\nabla w(0,s) \le \frac{1}{2}|y|^2\frac{A}{s},$$

$$(23) \qquad \left| w(y,s) - w(0,s) - y.\nabla w(0,s) - \frac{1}{2}y^T\nabla^2 w(0,s)y \right| \le \frac{1}{6}|y|^3\frac{A^{5/4}}{s^{3/2}},$$

$$(24) \qquad \left| \int \nabla w(y,s)\rho(y)dy - \nabla w(0,s) \right| \le C^*(N)\frac{A}{s},$$

$$(25) \qquad \text{and } |w(y,s) - w(0,s)| \le \frac{c_1}{\sqrt{s}}|y|$$

where $C^*(N) = \int |y|\rho(y)dy$.

Proof: By a Taylor expansion of $w(y,s)$ to the second order near $y = 0$, we write: $w(y,s) - w(0,s) - y.\nabla w(0,s) = \int_0^1 (1-t)y^T\nabla^2 w(ty,s)ydt$. Using (19) we get the first inequality.

The second and the forth inequalities are obtained in the same way by expanding $w(y,s)$ respectively until the third and the first order, and using $\|\nabla^3 w(s)\|_{L^\infty} \le \frac{A^{5/4}}{s^{3/2}}$ and $\|\nabla w(s)\|_{L^\infty} \le \frac{c_1}{\sqrt{s}}$.

For the third inequality, we write for all $y \in \mathbb{R}^N$, $\nabla w(y,s) - \nabla w(0,s) = y.\int_0^1 \nabla^2 w(ty,s)dt$. Using $\|\nabla^2 w(s)\|_{L^\infty} \le \frac{A}{s}$, we obtain $|\nabla w(y,s) - \nabla w(0,s)| \le |y|\frac{A}{s}$. Integrating this inequality with respect to ρdy, we get the conclusion. ■

Case 1: $w(s_n)$ **can not reach** $\kappa + \frac{c_0}{s_n}$

For all $\epsilon > 0$ and $A > 0$, there exists $S_1(A,\epsilon)$ such that if $s_n \ge S_1(A,\epsilon)$, Case 1 in Proposition 2.2 does not occur.

Proof: This estimate is in fact crucial and it follows from a blow-up argument. Assume that

$$(26) \qquad w(0,s_n) = \kappa + \frac{c_0}{s_n}.$$

Since $w(s_n) \in V_A(s_n)$, we have $\|w(s_n)\|_{L^\infty} \le \kappa + \frac{c_0}{s_n}$ and 0 is a global maximum for $w(s_n)$. Therefore, $\nabla w(0,s_n) = 0$. Hence, (22) yields
$w(y,s_n) \ge \kappa + \frac{c_0}{s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)|y|^2$ and
$\int w(y,s_n)\rho(y)dy \ge \kappa + \frac{c_0}{s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)\int |y|^2\rho(y)dy$
$= \kappa + \frac{c_0 - Nc_2}{s_n} - NC^*(N)\frac{A^{5/4}}{s_n^{3/2}} = \kappa + \frac{\epsilon}{s_n} - NC^*(N)\frac{A^{5/4}}{s_n^{3/2}} > \kappa$ for s_n large ($s_n \ge S_1(A,\epsilon) = \frac{2N^2C^*(N)^2A^{5/2}}{\epsilon^2}$).
This contradicts the global (in time) existence of w. Indeed, we have the following blow-up criterion for nonnegative solutions of (3):

Lemma 2.3 (A blow-up criterion for nonnegative solutions of (3)) *Consider $W \ge 0$ a solution of (3) and suppose that for some $s_0 \in \mathbb{R}$, $\int W(y,s_0)\rho(y)dy > \kappa$, then W blows-up in finite time $S > s_0$.*

Proof: See Proposition 3.5 in [16]. ■

Therefore, w blows-up in finite time S, which is a contradiction for $s_n \geq S_1(A,\epsilon)$.

Thus, Case 1 of Proposition 2.2 can not occur.

Case 2: $|\nabla w(s_n)|$ **can not reach** $\frac{c_1}{\sqrt{s_n}}$

There exist $\epsilon_2(p) > 0$ *such that* $\forall\epsilon \in (0,\epsilon_2(p))$, $\forall A > 0$, $\exists S_2(A,\epsilon)$ *such that if* $s_n \geq S_2(A,\epsilon)$, *then Case 2 in Proposition 2.2 can not occur.*

Proof: It follows from the bounds of $w(s_n)$ and $\nabla^2 w(s_n)$.
In this case, $|\nabla w(0,s_n)| = \frac{c_1}{\sqrt{s_n}}$. Using (22) with
$\hat{y}_n = (2\sqrt{p}+\epsilon)\sqrt{s_n}\frac{\nabla w(y_n,s_n)}{|\nabla w(y_n,s_n)|}$, we get:
$w(\hat{y}_n,s_n) \geq 0 + (2\sqrt{p}+\epsilon)\sqrt{s_n}\frac{c_1}{\sqrt{s_n}} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)(2\sqrt{p}+\epsilon)^2 s_n$
$= \kappa + 2p\epsilon + O\left(\epsilon^2\right) + O\left(\frac{1}{\sqrt{s_n}}\right)$ as $n \to +\infty$. Therefore, if $\epsilon \leq \epsilon_2(p)$ for some $\epsilon_2(p) > 0$, then $w(\hat{y}_n,s_n) \geq \kappa + p\epsilon + O\left(\frac{1}{\sqrt{s_n}}\right)$. Hence,

$$\kappa + \frac{c_0}{s_n} \geq \|w(s_n)\|_{L^\infty} \geq \kappa + p\epsilon + O\left(\frac{1}{\sqrt{s_n}}\right),$$

which is a contradiction if $s_n \geq S_2(A,\epsilon)$ for some $S_2(A,\epsilon)$.

Thus, Case 2 of Proposition 2.2 can not occur.

Case 3: $\varphi^T \int_{\mathbb{R}^N} \nabla^2 w(y,s_n)\rho(y)dy\varphi > -\frac{c_2}{s_n}$

$\forall\epsilon > 0$, $\forall A \geq 0$, $\exists S_3(A,\epsilon)$ *such that if* $s_n \geq S_3(A,\epsilon)$, *then Case 3 in Proposition 2.2 does not occur.*

Proof: We assume that $\varphi^T \int_{\mathbb{R}^N} \nabla^2 w(y,s_n)\rho(y)dy\varphi = -\frac{c_2}{s_n}$ for some unitary $\varphi \in \mathbb{R}^N$. We proceed in two steps: in Step 1, we derive a differential equation on $\int \nabla^2 w(y,s)\rho(y)dy$. In Step 2, we conclude the proof by a contradiction between this equation and the fact that w is globally defined in time.

Step 1: Equation on $\int \nabla^2 w(y,s)\rho(y)dy$

We recall that w is a solution of

$$\frac{\partial w}{\partial s} = \mathcal{L}w - \frac{p}{p-1}w + w^p \tag{27}$$

where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$ is a self-adjoint operator on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$ with ρ defined in (17). The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\text{spec}\,\mathcal{L} = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}.$$

Let us recall that in dimension 1, the eigenvalues are simple and the eigenfunction corresponding to $1 - \frac{m}{2}$ is

$$h_m(y) = \sum_{j=0}^{[\frac{m}{2}]} \frac{1}{j!(m-2j)!}(-1)^j y^{m-2j} \tag{28}$$

where h_m satisfies $\int h_m h_j \rho dy = \frac{2^m}{m!}\delta_{mj}$.
In dimension N, we write the spectrum of $\mathcal{L}$ as

$$\text{spec } \mathcal{L} = \{1 - \frac{m_1 + ... + m_N}{2} \mid m_1, ..., m_N \in \mathbb{N}\}. \tag{29}$$

For $(m_1, ..., m_N) \in \mathbb{N}$, the eigenfunction corresponding to $1 - \frac{m_1+...+m_N}{2}$ is

$$y \to h_{m_1}(y_1)...h_{m_N}(y_N). \tag{30}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of $L^2_\rho(\mathbb{R}^N)$, we can write

$$w(y,s) = w_0(s) + w_1(s).y + \left(\frac{1}{2}y^T w_2(s) y - trw_2(s)\right) + w_-(y,s) \tag{31}$$

where:
- $w_0(s) = \int w(y,s)\rho(y)dy \in \mathbb{R}$ (eigenvalue 1),
- $w_1(s) = \int w(y,s)\frac{y}{2}\rho(y)dy \in \mathbb{R}^N$ (eigenvalue $\frac{1}{2}$),
- $w_2(s) = \int w(y,s)M(y)\rho(y)dy$ is a $N \times N$ symmetric matrix (eigenvalue 0)

$$\text{with } M_{i.j}(y) = \frac{1}{4}y_i y_j - \frac{1}{2}\delta_{i,j}, \tag{32}$$

- $w_- = P_-(w)$ and P_- is the L^2_ρ projector on the negative subspace of $\mathcal{L}$.

Our purpose is to write an equation satisfied by $w_2(s)$. We claim the following:

Lemma 2.4 (Equation satisfied by $w_2(s)$**)** *For n large enough, we have:*
i) $w_1(s) = \int \nabla w(y,s)\rho(y)dy$ *and* $w_2(s) = \int \nabla^2 w(y,s)\rho(y)dy$,
ii) $|w_1(s_n)| \le \frac{c_1}{\sqrt{s_n}}$, $|w_2(s_n)| \le \frac{A}{s_n}$, $\forall y \in \mathbb{R}^N$, $|w_-(y,s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1 + |y|^3)$ *and* $\delta_0 \le w_0(s_n) \le \kappa$ *where* $\delta_0 = \frac{c_2^3}{128C(N)^2A^{5/4}}$ *for some* $C(N) > 0$.
iii)

$$\begin{aligned} w_2'(s_n) &= \left(pw_0(s_n)^{p-1} - \frac{p}{p-1}\right) w_2(s_n) \\ &+ p(p-1)w_0(s_n)^{p-2}\left[2w_2(s_n)^2 + w_1(s_n)\otimes w_1(s_n)\right] \\ &+ O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right). \end{aligned} \tag{33}$$

Proof: see Appendix A.
Remark: - If u and v are in $\mathbb{R}^N$, then we recall that $u \otimes v$ is the $N \times N$ matrix such that $(u \otimes v)_{i,j} = u_i v_j$ and $O(f)$ stands for a function which is bounded by $C(A,p,\epsilon)f$ as $n \to +\infty$.

Step 2: Conclusion for Case 3

Let $m(s) = \varphi^T w_2(s)\varphi$. Then, m is C^1, and since $w(s) \in V_A(s)$ for all $s \in [-\log T, s_n]$, we have: $m(s_n) = -\frac{c_2}{s_n}$ and $\forall s \in [-\log T, s_n]$, $m(s) \ge -\frac{c_2}{s}$. Thus,

$$m(s_n) = -\frac{c_2}{s_n} \text{ and } m'(s_n) \le \frac{c_2}{s_n^2}. \tag{34}$$

Multiplying (33) by φ^T on the left and φ on the right, we find:
$m'(s_n) = \left(pw_0(s_n)^{p-1} - \frac{p}{p-1}\right) m(s_n)$
$+p(p-1)w_0(s_n)^{p-2}\left[2m(s_n)^2 + (w_1(s_n).\varphi)^2\right] + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right).$
Therefore, since $(w_1(s_n).\varphi)^2 \geq 0$, we have
$pw_0(s_n)^{p-1} - \frac{p}{p-1} \geq \frac{-1}{m(s_n)}\Bigg(- m'(s_n) + 2p(p-1)w_0(s_n)^{p-2}m(s_n)^2$
$+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right)\Bigg).$
With (34), we obtain

$$w_0(s_n) \geq \left(\frac{1}{p-1} + \frac{s_n}{c_2}\left(- \frac{c_2}{ps_n^2} + 2(p-1)w_0(s_n)^{p-2}\frac{c_2^2}{s_n^2} + O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right) + O\left(\frac{1}{s_n^{5/2}}\right)\right)\right)^{\frac{1}{p-1}}. \tag{35}$$

Now, we claim that the following lemma yields the conclusion:

Lemma 2.5 *There exists positive constants $C(A,p,\epsilon)$ and $C'(A,p,\epsilon)$ such that*

$$|w(0,s_n) - \kappa| \leq \frac{C(A,p,\epsilon)}{s_n} \tag{36}$$

$$\textit{and} \qquad |w_1(s_n)| \leq \frac{C'(A,p,\epsilon)}{s_n}. \tag{37}$$

Indeed, if we inject (36) and (37) in (35), then we get
$w_0(s_n) \geq \left(\frac{1}{p-1} + \frac{s_n}{c_2}\left(-\frac{c_2}{ps_n^2} + 2(p-1)\kappa^{p-2}\frac{c_2^2}{s_n^2} + o\left(\frac{1}{s_n^2}\right)\right)\right)^{\frac{1}{p-1}}$, which yields
$w_0(s_n) \geq \kappa + 2\left(c_2 - \frac{\kappa}{2p}\right)\frac{1}{s_n} + o\left(\frac{1}{s_n}\right)$. Since $c_2 > \frac{\kappa}{2p}$, we obtain

$$w_0(s_n) > \kappa$$

for s_n large enough, which contradicts by lemma 2.3 the fact that w is globally defined on $[-\log T, +\infty)$.

Proof of lemma 2.5:
We derive from *ii*) of lemma 2.4 and (35): $w_0(s_n) \geq \left(\frac{1}{p-1} + O\left(\frac{1}{s_n}\right)\right)^{\frac{1}{p-1}} = \kappa + O\left(\frac{1}{s_n}\right)$. Since w is globally defined for $s \in [-\log T, +\infty)$, lemma 2.3 gives $w_0(s_n) \leq \kappa$. Hence,

$$w_0(s_n) = \kappa + O\left(\frac{1}{s_n}\right). \tag{38}$$

Integrating (22) with respect to ρdy, we obtain: $|w_0(s_n) - w(0,s_n)| \leq O\left(\frac{1}{s_n}\right)$. Together with (38), this gives (36).

Now, we claim that $|\nabla w(0,s_n)| \leq \frac{B}{s_n}$ with $B = \sqrt{2c_2(3c_0 + C(A,p,\epsilon))}$. Indeed, if not, then we use the left inequality of (22) and write for $\hat{y}_n = \frac{B}{c_2}\frac{\nabla w(0,s_n)}{|\nabla w(0,s_n)|}$:

$w(\hat{y}_n, s_n) \geq w(0, s_n) + \hat{y}_n.\nabla w(0, s_n) - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)|\hat{y}_n|^2$
$\geq \kappa - \frac{C(A,p,\epsilon)}{s_n} + \frac{B^2}{c_2 s_n} - \frac{1}{2}\left(\frac{c_2}{s_n} + C^*(N)\frac{A^{5/4}}{s_n^{3/2}}\right)\frac{B^2}{c_2^2}$
$= \kappa + \frac{3c_0}{s_n} + O\left(\frac{1}{s_n^{3/2}}\right)$. Therefore,

$$\kappa + \frac{c_0}{s_n} \geq \|w(s_n)\|_{L^\infty} \geq \kappa + \frac{2c_0}{s_n}$$

if s_n is large enough, which is a contradiction. Hence, $|\nabla w(0, s_n)| \leq \frac{B}{s_n}$. Using (24), we find $|w_1(s_n)| \leq \frac{C'(A,p,\epsilon)}{s_n}$ with $C'(A, p, \epsilon) = B + C^*(N)A$. This concludes the proof of lemma 2.5. ■

Thus, Case 3 can not occur.

Case 4: $\|\nabla^2 w(s_n)\|_{L^\infty}$ **can not reach** $\frac{A}{s_n}$

There exists $A_4(p)$ *such that for all* $A \geq A_4$*, and* $\epsilon > 0$*,* $\exists S_4(A, \epsilon)$ *such that if* $s_n \geq S_4(A, \epsilon)$*, then Case 4 of Proposition 2.2 can not occur.*

Proof: It follows from the bounds on w and $\nabla^3 w$. We have $|\nabla^2 w(0, s_n)| = \frac{A}{s_n}$. Hence, there exists $\eta_0 \in \{-1, 1\}$ and a unitary vector $\psi_n \in \mathbb{R}^N$ such that $\psi_n^T \nabla^2 w(0, s_n)\psi_n = \eta_0 \frac{A}{s_n}$. Let us notice that if $A > \frac{\kappa}{2p}$, then we have from (19) $\eta_0 = 1$ for n large enough.

Using (23) with $\hat{y}_n = \eta_1 \frac{\sqrt{s_n}}{A^{1/4}}\psi_n$ where $\eta_1 \in \{-1, 1\}$ is chosen so that $\hat{y}_n.\nabla w(0, s_n) \geq 0$, we write:
$w(\hat{y}_n, s_n) \geq w(0, s_n) + \hat{y}_n.\nabla w(0, s_n) + \frac{1}{2}\hat{y}_n^T \nabla^2 w(0, s_n)\hat{y}_n - \frac{1}{6}|\hat{y}_n|^3 \frac{A^{5/4}}{s_n^{3/2}}$
$\geq 0 + 0 + \frac{s_n}{2\sqrt{A}}\frac{A}{s_n} - \frac{s_n^{3/2}}{6A^{3/4}}\frac{A^{5/4}}{s_n^{3/2}} = \frac{\sqrt{A}}{3}$. If $A \geq 36\kappa^2$, then we have

$$\kappa + \frac{c_0}{s_n} \geq \|w(s_n)\|_{L^\infty} \geq 2\kappa$$

which is a contradiction for s_n large enough. ■

Thus, Case 4 can not occur.

Case 5: $\|\nabla^3 w(s_n)\|_{L^\infty}$ **can not reach** $\frac{A^{5/4}}{s_n^{3/2}}$

We first give a crucial uniform ODE comparison result for w in $V_A(s)$. Such a result has been shown in [16] for a fixed solution (see (6)). We claim that these estimates are in fact uniform for $w \in \hat{V}_A(s)$.

We have the following proposition:

Proposition 2.3 (ODE like behavior in $V_A(s)$**)** *For a given* $A > 0$*,* $\forall \eta > 0$*,* $\exists C_\eta > 0$ *such that for all* $s^* \geq -\log T$*, for all solution* w *of (3) defined for all* $s \geq -\log T$ *and satisfying* $w \in \hat{V}_A(s^*)$*, we have* $\forall x \in \mathbb{R}^N$*,* $\forall t \in [0, t^*]$*,*

$$\left|\frac{\partial u}{\partial t}(x,t) - u(x,t)^p\right| \leq \eta u(x,t)^p + C_\eta$$

where $t^* = T - e^{-s^*}$ *and* $u(x,t) = (T-t)^{\frac{1}{p-1}} w\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right)$.

Proof: It is mainly the same as in [16] (Theorem 3), and it uses a compactness procedure. See Appendix B. ∎

Now, we begin the treatment of Case 5.

We have

$$|\nabla^3 w(0,s_n)| = \frac{A^{5/4}}{s_n^{3/2}}, \text{ and } \forall s\in[-\log T, s_n],\ w(s)\in V_A(s). \tag{39}$$

Since $w(s_n)\in V_A(s_n)$, we have $0\le w(0,s_n)\le \kappa+\frac{c_0}{s_n}$. Therefore, we can assume that

$$w(0,s_n)\to a\in[0,\kappa] \text{ as } n\to+\infty.$$

We will consider the case where a is small in Part I, and let the case where it is not small for Part II. We first claim the following lemma:

Lemma 2.6 *$\forall S>0$, $\sup_{s\in[s_n-S,s_n]}|w(0,s)-\varphi_a(s-s_n)|\to 0$ as $n\to+\infty$ where φ_a is the solution of*

$$\begin{cases} \varphi_a'(s) &= -\frac{\varphi_a(s)}{p-1}+\varphi_a(s)^p\\ \varphi_a(0) &= a, \end{cases}$$

that is $$\varphi_a(s)=\kappa\left(1+\left(\frac{a^{1-p}}{p-1}-1\right)e^s\right)^{-\frac{1}{p-1}} \text{ if } a>0, \text{ and } \varphi_0(s)\equiv 0. \tag{40}$$

Proof: Let $z_n(s)=w(0,s)$, then we have from (3) $\forall s\in[s_n-S,s_n]$

$$\begin{cases} z_n'(s)+\frac{z_n(s)}{p-1}-z_n(s)^p=\Delta w(0,s)\\ z_n(s_n)\to a. \end{cases}$$

Since $\forall s\in[s_n-S,s_n]$, $w(s)\in V_A(s)$, we get $|\Delta w(0,s)|\le N\|\nabla^2 w(s)\|_{L^\infty}\le\frac{NA}{s}$. Hence, $\forall\eta>0$, we have for n large enough and $s\in[s_n-S,s_n]$:

$$\begin{cases} \left|z_n'(s)+\frac{z_n(s)}{p-1}-z_n(s)^p\right|\le\eta\\ |z_n(s_n)-a|\le\eta. \end{cases}$$

Therefore, by classical continuity arguments on ordinary differential equations, $w(0,s)=z_n(s)\to\varphi_a(s-s_n)$ as $n\to+\infty$, uniformly on $[s_n-S,s_n]$. This concludes the proof of lemma 2.6. ∎

Part I: Case where $a\le\delta(p)$

There exists $\delta(p)\in(0,\kappa)$ and $S_5(p)$ such that if $A\ge1$, $s_n\ge S_5(p)$ and $a\le\delta(p)$, then Case 5 of Proposition 2.2 can not occur.

This result follows from local estimates in new variables (ξ,τ) defined below and scaling arguments. We assume $a\le\delta(p)$ where $\delta(p)$ will be fixed later small enough, lower than $\frac{\kappa}{5}$.

Step 1: Setting of the problem:

For each $n\in\mathbb{N}$, we introduce $s_n'=\max\{\frac{s_n}{2}\}\cup\{s\in[\frac{s_n}{2},s_n]\mid w(0,s)\ge\frac{\kappa}{2}\}$. Let us remark that $w(0,s_n')\le\kappa$ and if $s_n'>\frac{s_n}{2}$, then $w(0,s_n')=\frac{\kappa}{2}$.

We have the following lemma:

Lemma 2.7 *There exists $S(\delta)\to+\infty$ as $\delta\to0$ such that for n large enough, $S(\delta)\le s_n-s_n'\le\frac{s_n}{2}$.*

Proof: Since $s'_n \geq \frac{s_n}{2}$, we have $s_n - s'_n \leq \frac{s_n}{2}$.
We get from (40) $S > 0$ such that $\forall s \in [-S, 0]$, $a \leq \varphi_a(s) \leq \frac{\kappa}{5}$ and $S \to +\infty$ as $a \to 0$. Hence, $S \to +\infty$ as $\delta \to 0$, since $a \leq \delta$.

Since $w(0,s) \to \varphi_a(s - s_n)$ as $n \to +\infty$ uniformly on $[s_n - S, s_n]$ by lemma 2.6, we obtain for n large enough $\forall s \in [s_n - S, s_n]$, $w(0,s) \leq \frac{\kappa}{4}$. Thus, $s'_n \leq s_n - S$. This concludes the proof of lemma 2.7. ■

Let us define for each $n \in \mathbb{N}$, $\xi \in \mathbb{R}^N$ and $\tau \in [-1, 1)$,

$$\begin{aligned} v_n(\xi,\tau) &= e^{-\frac{s'_n}{p-1}} u\left(\xi e^{-\frac{s'_n}{2}}, T + (\tau - 1)e^{-s'_n}\right) \\ &= (1-\tau)^{-\frac{1}{p-1}} w\left(\frac{\xi}{\sqrt{1-\tau}}, s'_n - \log(1-\tau)\right) \end{aligned} \tag{41}$$

where u is defined from w by (2) (take $a = 0$), and introduce $\tau_n \in [0,1]$ defined by $s'_n - \log(1-\tau_n) = s_n$. Then, v_n satisfies: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [-1,1)$

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + v_n^p. \tag{42}$$

From (39) and the definition of s'_n, we get: $v_n(\xi, 0) = w(\xi, s'_n)$,

$$\left\{ \begin{array}{rclrcl} v_n(0,0) & \leq & \frac{\kappa}{2}, & \|\nabla v_n(0)\|_{L^\infty} & \leq & \frac{c_1}{\sqrt{s'_n}}, \\ \|\nabla^2 v_n(0)\|_{L^\infty} & \leq & \frac{A}{s'_n}, & \|\nabla^3 v_n(0)\|_{L^\infty} & \leq & \frac{A^{5/4}}{s'^{3/2}_n}. \end{array} \right. \tag{43}$$

Note that if $s'_n > \frac{s_n}{2}$, then $v_n(0,0) = \frac{\kappa}{2}$.

Step 2: Estimates in v variable

We claim the following lemmas:

Lemma 2.8 (First estimate) *For n large enough, we have:*
i) $\forall \tau \in [-1, \tau_n]$, $\forall |\xi| \leq 2 s'^{1/4}_n$: $v_n(\xi,\tau) \leq C(p)$.
ii) For all $i = 1,2,3$, $\forall \tau \in [-\frac{1}{4}, \tau_n]$, $\forall |\xi| \leq \frac{3}{2} s'^{1/4}_n$, $|\nabla^i v_n(\xi,\tau)| \leq C'(p)$.

Lemma 2.9 (Refined estimate) *Assume that* $s'_n > \frac{s_n}{2}$. *Then,*
i) $\forall \tau \in [0, \tau_n]$, $\forall |\xi| \leq s'^{1/4}_n$, $\frac{\kappa}{4} \leq v_n(\xi,\tau) \leq C(p)$.
ii) There exist positive constants $C_6(p)$, $C_7(p)$ and $C_8(p)$ such that if $A \geq 1$ then $\forall \tau \in [0, \tau_n]$:

$$\forall |\xi| \leq \frac{s'^{1/4}_n}{4}, |\nabla v_n(\xi,\tau)| \leq \frac{C_6(p)}{\sqrt{s'_n}}, \tag{44}$$

$$\forall |\xi| \leq \frac{s'^{1/4}_n}{4^2}, |\nabla^2 v_n(\xi,\tau)| \leq \frac{C_7(p)A}{s'_n}, \tag{45}$$

$$\forall |\xi| \leq \frac{s'^{1/4}_n}{4^3}, |\nabla^3 v_n(\xi,\tau)| \leq \frac{C_8(p)A^{5/4}}{s'^{3/2}_n}. \tag{46}$$

Proof of lemma 2.8:
i) By Proposition 2.3, we have: $\forall \eta > 0$, $\forall x \in \mathbb{R}^N$, $\forall t \in [0, T - e^{-s_n})$

$$\left| \frac{\partial u}{\partial t}(x,t) - u(x,t)^p \right| \leq \eta u(x,t)^p + C_\eta.$$

Therefore, we get from (41): $\forall \eta > 0$, we have for n large enough: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [-1, \tau_n]$

$$\left| \frac{\partial v_n}{\partial \tau} - v_n(\xi,\tau)^p \right| \leq \eta v_n(\xi,\tau)^p + C_\eta e^{-\frac{p s'_n}{p-1}} \leq \eta \left(v_n(\xi,\tau)^p + 1 \right). \tag{47}$$

Using a Taylor expansion and (43), we get for n large enough: $\forall |\xi| \leq 2 s_n'^{1/4}$

$$|v_n(\xi,0) - v_n(0,0)| \leq \frac{2c_1}{s_n'^{1/4}} \text{ and } v_n(\xi,0) \leq \frac{3\kappa}{4}. \tag{48}$$

We take $\eta = \eta(p) > 0$ small enough such that $v_\eta(\tau)$ and $V_\eta(\tau)$ defined by

$$v_\eta(0) = V_\eta(0) = \frac{3\kappa}{4},\ v'_\eta = (1+\eta) v_\eta^p + \eta, \text{ and } V'_\eta = (1-\eta) V_\eta^p - \eta.$$

are well defined for all $\tau \in [-1, 1]$ and satisfy $\max\left(V_\eta(\tau), v_\eta(\tau)\right) \leq 2 v_0(1) = C(p)$.

Hence, for n large enough: $\forall |\xi| \leq 2 s_n'^{1/4}$,

$$\forall \tau \in [0, \tau_n],\ v_n(\xi,\tau) \leq v_\eta(\tau), \text{ and } \forall \tau \in [-1, 0],\ v_n(\xi,\tau) \leq V_\eta(\tau). \tag{49}$$

Therefore, $v_n(\xi,\eta) \leq C(p)$ for all $\tau \in [-1, \tau_n]$. This concludes the proof of i).

ii) We use a classical result (see Theorem 3 p. 406 in Friedman [8], see also Douglis and Nirenberg [5]):

Lemma 2.10 *Assume that h solves*

$$\frac{\partial h}{\partial \tau} = \Delta h + a(\xi,\tau) h$$

for $(\xi,\tau) \in D$ where $D = B(0,3) \times (-\tau_0, \tau_*)$ and $\tau_0, \tau_* \in [\frac{1}{2}, 1]$. Assume in addition that $\|a\|_{L^\infty} + |a|_{\alpha,D}$ is finite, where

$$|a|_{\alpha,D} = \sup_{(\xi,\tau),(\xi',\tau') \in D} \frac{|a(\xi,\tau) - a(\xi',\tau')|}{\left(|\xi - \xi'| + |\tau - \tau'|^{1/2}\right)^\alpha}$$

and $\alpha \in (0,1)$. Then,

$$\|h\|_{C^2(D')} + |\nabla^2 h|_{\alpha,D'} \leq K \|h\|_{L^\infty(D)}$$

where $K = K\left(\|a\|_{L^\infty(D)} + |a|_{\alpha,D}\right)$ and $D' = B(0,1) \times [-\tau_0 + \frac{1}{4}, \tau_*)$.

Since v_n is bounded on $B(0, 2 s_n'^{1/4}) \times [-1, \tau_n]$ (see i)), and since v_n and ∇v_n satisfy

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + a_1(\xi,\tau) v_n$$

and

$$\frac{\partial \nabla v_n}{\partial \tau} = \Delta\left(\nabla v_n\right) + a_2(\xi,\tau) \nabla v_n$$

for all $(\xi,\tau) \in B(0, 2 s_n'^{1/4}) \times [-1, \tau_n]$, with $a_2 = p a_1 = p v_n^{p-1}$, it is enough to prove that $|v_n|_{1, B(0, 2 s_n'^{1/4}) \times (-\frac{3}{4}, \tau_n)}$ is finite and to apply lemma 2.10 successively to v_n and ∇v_n in order to conclude the proof of ii).

For this purpose and from translation invariance, we restrict ourselves to $|\xi| < 3$ and write for all $(\xi, \tau) \in D = B(0,3) \times (-1, \tau_n)$, $v_n = h_1 + h_2$ where:
- h_1 is a solution of

$$\begin{cases} \frac{\partial h_1}{\partial \tau} &= \Delta h_1 \text{ for } (\xi,\tau) \in D \\ h_1(\xi,\tau) &= v_n(\xi,\tau) \text{ for } |\xi| = 3 \text{ and } \tau \in (-1,\tau_n) \\ h_1(\xi,-1) &= v_n(\xi,-1) \text{ for } |\xi| < 3, \end{cases}$$

- h_2 is a solution of

$$(50) \qquad \begin{cases} \frac{\partial h_2}{\partial \tau} &= \Delta h_2 + f(x,t) \text{ for } (\xi,\tau) \in \mathbb{R}^N \times (-1,\tau_n) \\ h_2(\xi,-1) &= 0 \text{ for all } \xi \in \mathbb{R}^N \end{cases}$$

with

$$(51) \qquad f(\xi,\tau) = v_n(\xi,\tau)^p 1_{\{(\xi,\tau)\in D\}} \le C(p).$$

From maximum principle, h_2 is bounded by $C(p)$ on $\mathbb{R}^N$, hence on D. Therefore, h_1 is bounded by $C(p)$ also. Applying lemma 2.10 with $h = h_1$ and $a = 0$, we see that in particular $|h_1|_{1,D'} \le C(p)$ where $D' = B(0,1) \times [-\frac{3}{4}, \tau_n)$.

We have from (50): $\forall (\xi,\tau) \in \mathbb{R}^N \times [-1,\tau_n)$,

$$(52) \qquad h_2(\xi,\tau) = \int_{-1}^{\tau} e^{(\tau-\sigma)\Delta} f(\sigma) d\sigma.$$

We claim that

$$(53) \qquad |h_2|_{1,\mathbb{R}^N\times[-1,\tau_n)} \le C(p),$$

which concludes the proof.

Proof of (53):

Let us recall that for all $\varphi \in L^\infty(\mathbb{R}^N)$: $\|e^{\tau\Delta}\varphi\|_{L^\infty} \le \|\varphi\|_{L^\infty}$,

$$(54) \qquad \|\nabla e^{\tau\Delta}\varphi\|_{L^\infty} \le \frac{C}{\sqrt{\tau}}\|\varphi\|_{L^\infty}, \text{ and } \|\frac{\partial}{\partial\tau} e^{\tau\Delta}\varphi\|_{L^\infty} \le \frac{C}{\tau}\|\varphi\|_{L^\infty}.$$

In order to prove (53), it is enough to estimate $|\nabla h_2(\xi,\tau)|$ and $\frac{|h_2(\xi,\tau_1)-h_2(\xi,\tau_2)|}{|\tau_1-\tau_2|^{1/2}}$ for all $\xi \in \mathbb{R}^N$ and $\tau, \tau_1, \tau_2 \in [-1,\tau_n)$.

By (52), (54) and (51), we have:
$|\nabla h_2(\xi,\tau)| = \left|\int_{-1}^{\tau} \nabla e^{(\tau-\sigma)\Delta} f(\sigma) d\sigma\right| \le \int_{-1}^{\tau} \frac{C}{\sqrt{\tau-\sigma}} \|f(\sigma)\|_{L^\infty} d\sigma$
$\le 2C(p)\sqrt{\tau+1} \le C(p)$.

Now, we take $\tau_2 < \tau_1$ and introduce $\tau_3 = \max(-1, \tau_2 - \sqrt{\tau_1-\tau_2})$. Then,
$\frac{|h(\xi,\tau_1)-h(\xi,\tau_2)|}{\sqrt{\tau_1-\tau_2}} = (\tau_1-\tau_2)^{-\frac{1}{2}} \left|\int_{-1}^{\tau_1} e^{(\tau_1-\sigma)\Delta} f(\sigma)d\sigma - \int_{-1}^{\tau_2} e^{(\tau_2-\sigma)\Delta} f(\sigma) d\sigma\right|$
$\le I + II + III$ with $I = (\tau_1-\tau_2)^{-\frac{1}{2}} \int_{\tau_3}^{\tau_1} \|e^{(\tau_1-\sigma)\Delta} f(\sigma)\|_{L^\infty} d\sigma$,
$II = (\tau_1-\tau_2)^{-\frac{1}{2}} \int_{\tau_3}^{\tau_2} \|e^{(\tau_2-\sigma)\Delta} f(\sigma)\|_{L^\infty} d\sigma$ and
$III = \int_{-1}^{\tau_3} \left|e^{(\tau_1-\sigma)\Delta} f(\sigma) - e^{(\tau_2-\sigma)\Delta} f(\sigma)\right| d\sigma$.

From (54) and (51), we have:
$I \le (\tau_1-\tau_2)^{-\frac{1}{2}} \int_{\tau_3}^{\tau_1} C(p) d\sigma = C(p)(\tau_1-\tau_2)^{-\frac{1}{2}}(\tau_1-\tau_3)$
$\le C(p)(\tau_1-\tau_2)^{-\frac{1}{2}}(\tau_1-\tau_2+\sqrt{\tau_1-\tau_2}) \le C(p)$.
Similarly, $II \le C(p)$. ■

For III, we write
$III = (\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\left|\int_{\tau_2-\sigma}^{\tau_1-\sigma}\frac{\partial}{\partial\sigma_1}e^{\sigma_1\Delta}f(\sigma)d\sigma_1\right|$
$\le (\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\int_{\tau_2-\sigma}^{\tau_1-\sigma}\frac{C}{\sigma_1}d\sigma_1$ by (54),
$\le (\tau_1-\tau_2)^{-\frac{1}{2}}\int_{-1}^{\tau_3}d\sigma\frac{C(\tau_1-\tau_2)}{(\tau_2-\sigma)}$
$\le C(\tau_1-\tau_2)^{\frac{1}{2}}(\tau_3+1)(\tau_2-\tau_3)^{-1}\le C(\tau_1-\tau_2)^{\frac{1}{2}}\times 2\times(\sqrt{\tau_1-\tau_2})^{-1}=C.$
Thus, $\forall\xi\in\mathbb{R}^N$, $\forall\tau_1,\tau_2\in[-1,\tau_n)$,
$|h_2(\xi,\tau_1)-h_2(\xi,\tau_2)|\le C|\tau_1-\tau_2|^{\frac{1}{2}}$.
This concludes the proof of (53) and the proof of lemma 2.8 also. ■

Proof of lemma 2.9:
In this case, $v_n(0,0)=\frac{\kappa}{2}$.
$i)$ As in lemma 2.6, (47) and (48) yield $\sup_{|\xi|\le s_n'^{1/4},\tau\in[0,\tau_n]}|v_n(\xi,\tau)-v(\tau)|\to 0$ as $n\to+\infty$, where v is the solution of

$$v'(\tau)=v(\tau)^p,\ v(0)=\frac{\kappa}{2},\ \text{that is } v(\tau)=\kappa\left(2^{p-1}-\tau\right)^{-\frac{1}{p-1}}.$$

Since $\forall\tau\in[0,1]$, $v(\tau)\ge\frac{\kappa}{2}$, we have for n large enough:

$$\forall|\xi|\le s_n'^{1/4},\ \forall\tau\in[0,1),\ \frac{\kappa}{4}\le v_n(\xi,\tau). \qquad (55)$$

$i)$ of lemma 2.8 yields the upper bound.

$ii)$ Let us recall the following lemma:

Lemma 2.11 *Assume that* $z(\xi,\tau)$ *satisfies* $\forall|\xi|\le 4B_1$, $\forall\tau\in[0,\tau_*]$:

$$\begin{cases}\frac{\partial z}{\partial\tau}\le\Delta z+\lambda z+\mu,\\ z(\xi,0)\le z_0,\ z(\xi,\tau)\le B_2\end{cases}\qquad (56)$$

where $\tau_*\le 1$. *Then,* $\forall|\xi|\le B_1$, $\forall\tau\in[0,\tau_*]$,

$$z(\xi,\tau)\le e^{\lambda\tau}\left(z_0+\mu+CB_2e^{-\frac{B_1^2}{4}}\right).$$

Proof: See Appendix C. ■

Estimate on $\nabla v_n(\xi,\tau)$:
We estimate $h(\xi,\tau)=|\nabla v_n(\xi,\tau).\alpha|$ where α is a unitary vector of $\mathbb{R}^N$.
From (42), Kato's inequality, (43) and lemma 2.8, we see that $\forall|\xi|\le s_n'^{1/4}$, $\forall\tau\in[0,\tau_n]$,

$$\begin{cases}\frac{\partial h}{\partial\tau}\le\Delta h+pv_n^{p-1}h\le\Delta h+pC(p)^{p-1}h,\\ h(\xi,0)\le\frac{c_1}{\sqrt{s_n'}},\ h(\xi,\tau)\le C'(p).\end{cases}\qquad (57)$$

Using lemma 2.11, we get: $\forall|\xi|\le\frac{s_n'^{1/4}}{4}$, $\forall\tau\in[0,\tau_n]$,

$$h(\xi,\tau)\le e^{pC(p)^{p-1}}\left(\frac{c_1}{\sqrt{s_n'}}+CC'(p)e^{-\frac{s_n'^{1/2}}{4}}\right)$$

which yields (44) since $c_1\le\frac{\kappa}{\sqrt{p}}+2\sqrt{p}$.

Estimate on $\nabla^2 v_n(\xi,\tau)$:
We estimate $\theta(\xi,\tau) = |\alpha^T \nabla^2 v_n(\xi,\tau)\alpha|$ where α is a unitary vector in $\mathbb{R}^N$.

From (42) and Kato's inequality, we have: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0,1)$,

$$\frac{\partial \theta}{\partial \tau} \le \Delta\theta + p v_n^{p-1}\theta + p(p-1)v_n^{p-2}|\nabla v_n|^2.$$

Using (44), lemma 2.8, $i)$ of lemma 2.9 and (43), we claim that $\forall |\xi| \le \frac{s_n'^{1/4}}{4}$, $\forall \tau \in [0,\tau_n]$,

$$\begin{cases} \frac{\partial\theta}{\partial\tau} \le \Delta\theta + C(p)\theta + C(p)\frac{C_6(p)^2}{s_n'}, \\ \theta(\xi,0) \le \frac{A}{s_n'},\ \theta(\xi,\tau) \le C'(p) \end{cases}$$

By lemma 2.11, we obtain, $\forall |\xi| \le \frac{s_n'^{1/4}}{4^2}$, $\forall \tau \in [0,\tau_n]$,

$$\theta(\xi,\tau) \le e^{C(p)}\left(\frac{A}{s_n'} + C(p)\frac{C_6(p)^2}{s_n'} + CC'(p)e^{-\frac{s_n'^{1/2}}{4^3}}\right).$$

Since $A \ge 1$, this yields (45).

Estimate on $\nabla^3 v_n(\xi,\tau)$:
We estimate $\nu(\xi,\tau) = |\nabla^3 v_n(\xi,\tau)(\alpha,\beta,\gamma)|$ where α, β and γ are unitary vectors in $\mathbb{R}^N$.

From (42) and Kato's inequality, we have: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0,1)$,

$$\frac{\partial \nu}{\partial \tau} \le \Delta\nu + p v_n^{p-1}\nu + 3p(p-1)v_n^{p-2}|\nabla v_n||\nabla^2 v_n| + p(p-1)|p-2|v_n^{p-3}|\nabla v_n|^3.$$

Using (44), (45), lemma 2.8, $i)$ of lemma 2.9 and (43), we get: $\forall |\xi| \le \frac{s_n'^{1/4}}{4^2}$, $\forall \tau \in [0,\tau_n]$.

$$\begin{cases} \frac{\partial\nu}{\partial\tau} \le \Delta\nu + C(p)\nu + C(p)\frac{\left(C_6(p)^3 + C_6(p)C_7(p)\right)}{s_n'^{3/2}}, \\ \nu(\xi,o) \le \frac{A^{5/4}}{s_n'^{3/2}}, \nu(\xi,\tau) \le C'(p). \end{cases}$$

Applying again lemma 2.11, we obtain: $\forall |\xi| \le \frac{s_n'^{1/4}}{4^3}$, $\forall \tau \in [0,1)$,

$$\nu(\xi,\tau) \le e^{C(p)}\left(\frac{A^{5/4}}{s_n'^{3/2}} + C(p)\frac{\left(C_6(p)^3 + C_6(p)C_7(p)\right)}{s_n'^{3/2}} + CC'(p)e^{-\frac{s_n'^{1/2}}{4^5}}\right).$$

Since $A \ge 1$, this yields (46).

This concludes the proof of lemma 2.9. ∎

Step 3: Conclusion of the proof

From (41), we have

$$\nabla^3 w(0,s_n) = (1-\tau_n)^{\left(\frac{1}{p-1}+\frac{3}{2}\right)}\nabla^3 v_n(0,\tau_n), \tag{58}$$

where τ_n is defined by $s_n' - \log(1-\tau_n) = s_n$.

- If $s_n' = \frac{s_n}{2}$, then $1-\tau_n = e^{s_n'-s_n} = e^{-\frac{s_n}{2}}$. Hence, (58) and lemma 2.8 yield:

$$|\nabla^3 w(0,s_n)| \le e^{-\frac{s_n}{2}\left(\frac{1}{p-1}+\frac{3}{2}\right)}C'(p).$$

This contradicts (39) for s_n large enough.

- If $s'_n > \frac{s_n}{2}$, then we have by lemma 2.7 $s'_n - s_n \leq -S(\delta)$ for n large enough. Therefore, (58) and lemma 2.9 yield

$$
\begin{aligned}
(59) \qquad |s_n^{3/2}\nabla^3 w(0,s_n)| &\leq A^{5/4}C_8(p)e^{(s'_n-s_n)\left(\frac{1}{p-1}+\frac{3}{2}\right)}\left(\frac{s_n}{s'_n}\right)^{\frac{3}{2}}\\
&\leq A^{5/4}C_8(p)e^{-S(\delta)\left(\frac{1}{p-1}+\frac{3}{2}\right)}\left(\frac{s_n}{s_n/2}\right)^{3/2}.
\end{aligned}
$$

Since $S(\delta)\to+\infty$ as $\delta\to 0$, we fix $\delta(p)>0$ such that

$$C_8(p)e^{-S(\delta)\left(\frac{1}{p-1}+\frac{3}{2}\right)}2^{3/2}\leq\frac{1}{2}.$$

Therefore, (59) yields $|s_n^{3/2}\nabla^3 w(0,s_n)|\leq\frac{A^{5/4}}{2}$. This contradicts (39).

Thus, Case 5 can not occur if $a\leq\delta(p)$.

Part II: Case where $a\geq\delta(p)$

There exists $A_6(p)>0$ and $S_6(p)$ such that for all $A\geq A_6(p)$, if $s_n\geq S_6(p)$, then Case 5 of Proposition 2.2 can not occur if $a\geq\delta(p)$.

This follows from linear estimates on w, for the spectrum of the linear part of the equation on $\nabla^3 w$ is fully negative.

Let us remark that in this case, we have:

$$(60) \qquad \forall s\in[s_n-1,s_n],\ \forall|y|\leq\frac{\delta\sqrt{s}}{4c_1},\ \frac{\delta}{4}\leq w(y,s)\leq\kappa+1.$$

Indeed, the upper bound follows from the fact that $w(s)\in V_A(s)$. For the lower bound, we notice that since $a\geq\delta$, we have from lemma 2.6 and (40): $\forall s\in[s_n-1,s_n]$, $w(0,s)\geq\frac{\delta}{2}$ for s_n large enough. Therefore, we have by (25): $w(y,s)\geq w(0,s)-\frac{c_1}{\sqrt{s}}|y|\geq\frac{\delta}{2}-\frac{\delta\sqrt{s}}{4c_1}\frac{c_1}{\sqrt{s}}=\frac{\delta}{4}$.

From (39), we have the existence of $\alpha,\beta,\gamma\in\mathbb{R}^N$ such that $|\alpha|=|\beta|=|\gamma|=1$ and

$$(61) \qquad |\nabla^3 w(0,s_n)(\alpha,\beta,\gamma)|=\frac{A^{5/4}}{s_n^{3/2}}.$$

Our strategy is to derive from (3) an equation on $g(y,s)=\nabla^3 w(y,s)(\alpha,\beta,\gamma)$ and to do a priori estimates on it in order to contradict (61). We in fact define

$$(62)\ G(y,s)=F(y,s)\chi(y,s),\qquad F(y,s)=|g(y,s)|=|\nabla^3 w(y,s)(\alpha,\beta,\gamma)|,$$

$$(63) \qquad \chi(y,s)=\chi_0\left(\frac{8c_1|y|}{\delta\sqrt{s}}\right)$$

and $\chi_0\in C^\infty([0,+\infty),\mathbb{R}^+)$ satisfies $\chi_0(z)=1$ for $|z|\leq 1$, $\chi_0(z)=0$ for $|z|\geq 2$.

From (3), we see that

$$
\begin{aligned}
\frac{\partial g}{\partial s} &= \left(\mathcal{L}-\frac{3}{2}+pw(y,s)^{p-1}-\frac{p}{p-1}\right)g\\
&+ p(p-1)w(y,s)^{p-2}\big((\alpha.\nabla w)(\beta^T\nabla^2 w\gamma)+(\beta.\nabla w)(\gamma^T\nabla^2 w\alpha)\\
&+ (\gamma.\nabla w)(\alpha^T\nabla^2 w\beta)\big)\\
&+ p(p-1)(p-2)w(y,s)^{p-3}(\alpha.\nabla w)(\beta.\nabla w)(\gamma.\nabla w).
\end{aligned}
$$

We see from (39) and definition 2.1 that for s_n large enough, we have:
- $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$, $pw(y,s)^{p-1} - \frac{p}{p-1} \le \frac{1}{4}$,
- $0 < \frac{\kappa}{\sqrt{p}} \le c_1(\epsilon) \le \frac{\kappa}{\sqrt{p}} + 2\sqrt{p}$ for all $\epsilon \in (0,1)$.

Therefore, F satisfies the following inequality: $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$,

$$\frac{\partial F}{\partial s} \le (\mathcal{L} - \frac{5}{4})F + \frac{C(p)A}{s^{3/2}} w(y,s)^{p-2} + \frac{C(p)}{s^{3/2}} w(y,s)^{p-3}.$$

Hence, by (63) and (60), G satisfies the following inequality: $\forall y \in \mathbb{R}^N$, $\forall s \in [s_n - 1, s_n]$

$\frac{\partial G}{\partial s} \le (\mathcal{L} - \frac{5}{4})G + \frac{C(p)A}{s^{3/2}} \chi w(y,s)^{p-2} + \frac{C(p)}{s^{3/2}} \chi w(y,s)^{p-3} + F(\frac{\partial \chi}{\partial s} + \Delta\chi + \frac{1}{2} y.\nabla\chi) - 2\nabla.(F\nabla\chi).$

$\le (\mathcal{L} - \frac{5}{4})G + C(p)\frac{(A+1)}{s^{3/2}} + C(p)\frac{A^{5/4}}{s^{3/2}} 1_{\{|y| \ge \frac{\delta\sqrt{s}}{8c_1}\}} - 2\nabla.(F\nabla\chi).$

Using an integral formulation of this inequality between $s_n - \eta$ and s_n where $\eta(p)$ is fixed such that

$$(64)\qquad \eta \in (0,1) \text{ and } \frac{\delta^2}{512 c_1^2 (1 - e^{-\eta})} \ge \frac{\delta^2}{512 c_1(1)^2 (1 - e^{-\eta})} > \frac{1}{4},$$

we obtain

$$(65)\qquad G(0, s_n) \le I + II + III + IV$$

where

$I = \left[e^{\eta(\mathcal{L} - \frac{5}{4})} G(s_n - \eta) \right](0)$,

$II = \left[\int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} C(p) \frac{(A+1)}{t^{3/2}} \right](0)$,

$III = \left[\int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} C(p) \frac{A^{5/4}}{t^{3/2}} 1_{\{|x| \ge \frac{\delta\sqrt{t}}{8c_1}\}} \right](0)$ and

$IV = \left[-2 \int_{s_n - \eta}^{s_n} dt e^{(s_n - t)(\mathcal{L} - \frac{5}{4})} \nabla.(F\nabla\chi) \right](0)$.

Let us recall that the kernel of $\mathcal{L}$ is: $\forall s > 0$,

$$(66)\qquad e^{s(\mathcal{L} - \frac{5}{4})}(y,x) = \frac{e^{-\frac{s}{4}}}{(4\pi(1 - e^{-s}))^{N/2}} \exp\left(-\frac{|ye^{-\frac{s}{2}} - x|^2}{4(1 - e^{-s})} \right)$$

and that for all $\varphi \in L^\infty(\mathbb{R}^N)$,

$$(67)\quad \|e^{s(\mathcal{L} - \frac{5}{4})}\varphi\|_{L^\infty} \le e^{-\frac{s}{4}} \|\varphi\|_{L^\infty},\ \|e^{s(\mathcal{L} - \frac{5}{4})}\nabla\varphi\|_{L^\infty} \le \frac{C(N)}{\sqrt{1 - e^{-s}}} \|\varphi\|_{L^\infty}.$$

From (67), (62) and (39), we have

$I \le e^{-\frac{\eta}{4}} \|G(s_n - \eta)\|_{L^\infty} \le e^{-\frac{\eta}{4}} \frac{A^{5/4}}{(s_n - \eta)^{3/2}}$.

Again, by (67), we have

$II \le \int_{s_n - \eta}^{s_n} dt e^{-\frac{(s_n - t)}{4}} C(p) \frac{(A+1)}{t^{3/2}} \le C(p) \frac{(A+1)}{(s_n - \eta)^{3/2}} \eta \le C(p) \frac{(A+1)}{(s_n - \eta)^{3/2}}$ by (64).

By (66), we have:

$III = \int_{s_n - \eta}^{s_n} dt \frac{e^{-\frac{(s_n - t)}{4}}}{\left(4\pi(1 - e^{-(s_n - t)})\right)^{N/2}} \int_{\{|x| \ge \frac{\delta\sqrt{t}}{8c_1}\}} dx \exp\left(-\frac{|x|^2}{4(1 - e^{-(s_n - t)})} \right) C(p) \frac{A^{5/4}}{t^{3/2}}$.

For $|x| \ge \frac{\delta\sqrt{t}}{8c_1}$ and $t \in [s_n - \eta, s_n]$, we have

$\exp\left(-\frac{|x|^2}{4(1 - e^{-(s_n - t)})} \right) = \exp\left(-\frac{|x|^2}{8(1 - e^{-(s_n - t)})} \right) \exp\left(-\frac{|x|^2}{8(1 - e^{-(s_n - t)})} \right)$

$\leq \exp\left(-\frac{\delta^2 t}{512c_1^2(1-e^{-\eta})}\right)\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right) \leq e^{-\frac{t}{4}}\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)$ from (64).
Therefore,
$III \leq C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\int_{s_n-\eta}^{s_n}dt\int\frac{dx}{\left(4\pi(1-e^{-(s_n-t)})\right)^{N/2}}\exp\left(-\frac{|x|^2}{8(1-e^{-(s_n-t)})}\right)$
$= C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\int_{s_n-\eta}^{s_n}dt\int dXe^{-|X|^2}$
$= C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}\eta \leq C(p)\frac{A^{5/4}e^{-\frac{s_n}{4}}}{(s_n-\eta)^{3/2}}$ by (64).

From (66) and integration by parts, we have:
$IV \leq C(N)\int_{s_n-\eta}^{s_n}\frac{dt}{\sqrt{1-e^{-s_n-t}}}\|F(t)\nabla\chi(t)\|_{L^\infty}$. From (62), (39) and (63), we have $F(x,t) \leq \frac{A^{5/4}}{t^{3/2}}$ and $|\nabla\chi| \leq \frac{C}{\sqrt{t}}$. Therefore,
$IV \leq \frac{CA^{5/4}}{(s_n-\eta)^2}\int_{s_n-\eta}^{s_n}\frac{dt}{\sqrt{1-e^{-s_n-t}}} \leq C\frac{A^{5/4}}{(s_n-\eta)^2}C\sqrt{\eta} \leq C\frac{A^{5/4}}{(s_n-\eta)^2}$ by (64).

From (65) and (62), we then get:
$|g(0,s_n)| = G(0,s_n) \leq \frac{A^{5/4}}{(s_n-\eta)^{3/2}}\left(e^{-\frac{\eta}{4}} + C(p)e^{-\frac{s_n}{4}} + \frac{C}{\sqrt{s_n-\eta}}\right) + C(p)\frac{(A+1)}{(s_n-\eta)^{3/2}}$.

Now, we take $A \geq A_5(p)$ such that $C(p)(A+1) \leq \left(e^{-\frac{\eta}{5}} - e^{-\frac{\eta}{4}}\right)A^{5/4}$, and $s_n \geq S_5(p)$ such that
$\frac{1}{(s_n-\eta)^{3/2}}\left(e^{-\frac{\eta}{5}} + C(p)e^{-\frac{s_n}{4}} + \frac{C}{\sqrt{s_n-\eta}}\right) \leq \frac{e^{-\frac{\eta}{6}}}{s_n^{3/2}}$. If $s_n \geq S_5(p)$, then we have:
$|\nabla^3 w(0,s_n)(\alpha,\beta,\gamma)| = |g(0,s_n)| \leq e^{-\frac{\eta}{6}}\frac{A^{5/4}}{s_n^{3/2}} < \frac{A^{5/4}}{s_n^{3/2}}$. This contradicts (61).

Thus, Case 5 can not occur if $a \geq \delta(p)$.

3 Blow-up profile notions for equation (1)

In this section, we prove Theorems 2 and 3.
Let us first show the existence of a profile in the intermediate variable $z = \frac{y}{\sqrt{s}}$.

Proof of Theorem 2:
The theorem is a consequence of:
- the behavior of the solution $w(y,s)$ for y bounded,
- the pointwise estimates on $\Delta w(y,s)$ in Theorem 1, which will enable us to treat this term in equation (3) as a perturbation.

Let $u(t)$ be a solution of (1) which blows-up at time $T > 0$ and satisfies $u(0) \in H^1(\mathbb{R}^N)$. Let x_0 be a blow-up point of $u(t)$ and consider w_{x_0} defined by (2). We just write w for w_{x_0}.

The proof is in two steps:

Step 1: Reduction of the problem
According to Filippas and Liu [7] and Velázquez [18],
- either $\forall R > 0$, $\sup_{|y|\leq R}|w(y,s) - \kappa| \leq C(R)e^{-\delta s}$ for some $\delta > 0$,
- or there exists $k \in \{0,...,N-1\}$ and a $N \times N$ orthonormal matrix Q such

that $\forall R>0,\ \sup\limits_{|y|\le R}\left|w(y,s)-\left[\kappa+\frac{\kappa}{2ps}\left((N-k)-\frac{1}{2}y^TA_ky\right)\right]\right|=o\left(\frac{1}{s}\right)$
as $s\to+\infty$ where

$$A_k=Q\begin{pmatrix} I_{N-k} & 0\\ 0 & 0\end{pmatrix}Q^{-1} \tag{68}$$

and I_{N-k} is the $(N-k)\times(N-k)$ identity matrix.
By direct calculations, we summarize both cases by:

$$\forall R>0,\ \sup_{|y|\le R}|w(y,s)-f_k\left(\frac{y}{\sqrt{s}}\right)-\frac{a}{s}|=o\left(\frac{1}{s}\right) \tag{69}$$

where

$$f_k(z)=\left(p-1+\frac{(p-1)^2}{4p}z^TA_kz\right)^{-\frac{1}{p-1}},\ a=\frac{\kappa(N-k)}{2p}, \tag{70}$$

$k\in\{0,1,...,N\}$ and A_k is defined in (68) (take $A_N=0$).

We claim now that (69) implies that the convergence is uniform on larger sets:

Proposition 3.1 (Convergence extension to space-time parabolas) *Assume that w is a solution of (3) which satisfies (69). Then, $\forall K_0>0$,*

$$\sup_{|z|\le K_0}|w(z\sqrt{s},s)-f_k(z)|\to 0 \ \textit{as}\ s\to+\infty.$$

It is immediate that Theorem 2 is a direct consequence or Proposition 3.1. Thus, we now focus on the proof of Proposition 3.1.

The main feature in the proof is an a priori estimate on

$$q(y,s)=w(y,s)-f_k(\frac{y}{\sqrt{s}}). \tag{71}$$

We consider the equation satisfied by q as a perturbation of a hyperbolic equation (the size of the perturbation is crucially controlled by Theorem 1). We claim the following result:

Proposition 3.2 (Hyperbolic estimate on $q(y,s)$ for $A\le|y|\le K_0\sqrt{s}$) *Assume (69). Then, for any $K_0>0$, there exist $A_0(K_0)>0$ and $B(K_0)>0$ such that for all $A\ge A_0$, there exists $S_0(K_0,A)$ with the following property: If $\omega\in S^{N-1}$, $s_0\ge S_0$, then*

$$\forall s\in[s_0,s_1],\ |q(Ae^{\frac{s-s_0}{2}}\omega,s)|\le B\frac{e^{s-s_0}}{s_0}$$

where $s_1\ge s_0$ is defined by

$$Ae^{\frac{s_1-s_0}{2}}=K_0\sqrt{s_1}. \tag{72}$$

Let us first show how this proposition concludes the proof of Proposition 3.1.
Remark: We notice that it directly follows from Proposition 3.2 that for S_0 larger, we have

$$\forall s\in[s_0,s_1],\ |q(Ae^{\frac{s-s_0}{2}}\omega,s)|\le\frac{2BK_0^2}{A^2}. \tag{73}$$

Indeed, we have $\forall s \in [s_0, s_1]$, $Ae^{\frac{s-s_0}{2}} \le K_0\sqrt{s} \le K_0\sqrt{s_1}$. Therefore, $|q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le B\frac{e^{s-s_0}}{s_0} \le \frac{BK_0^2}{A^2}\frac{s_1}{s_0}$. If K_0 and A are fixed, then it is easy to see that $s_1 \sim s_0$ as $s_0 \to +\infty$. One might take $S_0(K_0, A)$ larger to have $\frac{s_1}{s_0} \le 2$, which yields $|q(Ae^{\frac{s-s_0}{2}}\omega, s)| \le \frac{2BK_0^2}{A^2}$.
We now prove Proposition 3.1.
Let $K_0 > 0$ and $\epsilon > 0$. Fix $A \ge A_0(K_0)$ so that $\frac{2BK_0^2}{A^2} \le \epsilon$.
By (69), there exists $s_{02}(\epsilon)$ such that

$$\forall s \ge s_{02},\ \forall |y| \le A, |q(y, s)| \le \epsilon. \tag{74}$$

Let $s_{03}(K_0, A) \ge S_0$ be defined by $Ae^{\frac{s_{03}-S_0}{2}} = K_0\sqrt{s_{03}}$. We claim that $\forall s \ge \max(s_{02}(\epsilon), s_{03}(K_0, A))$, $\forall |y| \le K_0\sqrt{s}$, $|q(y, s)| \le \epsilon$.
Indeed, if $|y| \le A$, then the conclusion follows from (74). If $A \le |y| \le K_0\sqrt{s}$, we define $s_0(|y|, s)$ by $|y| = Ae^{\frac{s-s_0}{2}}$. By construction of $s_{03}(K_0, A)$, we have $s_0(|y|, s) \ge S_0(K_0, A)$. We also have $s_0 \le s \le s_1$, since $Ae^{\frac{s-s_0}{2}} = |y| \le K_0\sqrt{s}$ and $Ae^{\frac{s_1-s_0}{2}} = K_0\sqrt{s_1}$. Applying the remark (73) coming after Proposition 3.2 gives $|q(y, s)| = |q(Ae^{\frac{s-s_0}{2}}\frac{y}{|y|}, s)| \le \frac{2BK_0^2}{A^2} \le \epsilon$. This is the conclusion of Proposition 3.1 and that of Theorem 2 also. Let us now prove proposition 3.2.

Step 2: Hyperbolic estimates: *Proof of Proposition 3.2*:
Define

$$B(K_0) = 3(|a| + 1 + C_4)\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}} \tag{75}$$

with $C_4 = C_5 + \frac{1}{2}\|z.\nabla f(z)\|_{L^\infty}$, C_5 is the constant given by Theorem 1 such that $\|\Delta w(s)\|_{L^\infty} \le \frac{C_5}{s}$ and a is defined in (70).

We consider $A \ge A_0(K_0)$ and $s_0 \ge S_0(K_0, A)$ ($A_0(K_0)$ and $S_0(K_0, A)$ will be defined later).

Let $\omega \in S^{N-1}$ and introduce

$$y(A, \omega, s_0, s) = Ae^{\frac{s-s_0}{2}}\omega \text{ and } h(A, \omega, s_0, s) = q\left(y(A, \omega, s_0, s), s\right). \tag{76}$$

For simplicity, we will just write $y(s)$ and $h(s)$. Let us define $s_{04}(K_0, A)$ (independent of w) such that $\forall s_0 \ge s_{04}(K_0, A)$, s_1 (introduced in (72)) is well defined and satisfies $s_1 \le 2s_0$, and

$$|h(s_0)| = |q(A\omega, s_0)| \le \frac{|a| + 1}{s_0} < \frac{B}{s_0} \tag{77}$$

by definition of $B(K_0)$ (This follows directly from (69)).

The proof of Proposition 3.2 reduces now to prove that $\forall s_0 \ge S_0(K_0, A)$, $\forall s \in [s_0, s_1]$, $|h(s)| \le B\frac{e^{s-s_0}}{s_0}$. We proceed by a priori estimates.

We suppose by contradiction the existence of some $s_* \in [s_0, s_1]$ such that

$$\forall s \in [s_0, s_*),\ |h(s)| < \frac{Be^{s-s_0}}{s_0} \text{ and } |h(s_*)| = \frac{Be^{s_*-s_0}}{s_0}. \tag{78}$$

Since f_k is a solution of $0 = -\frac{1}{2}y.\nabla f_k(z) - \frac{f_k(z)}{p-1} + f_k(z)^p$, we derive from (71) and (3) an equation satisfied by q: $\forall y \in \mathbb{R}^N$, $\forall s \ge -\log T$:

$$\frac{\partial q}{\partial s} = -\frac{1}{2}y.\nabla q + \left(pf_k\left(\frac{y}{\sqrt{s}}\right) - \frac{1}{p-1}\right)q + N(q) + r(y, s)$$

where $N(q) = (f_k + q)^p - f_k^p - pf_k^{p-1}q$ and $r(y,s) = \frac{1}{2}\frac{y}{s^{3/2}}\nabla f_k\left(\frac{y}{\sqrt{s}}\right) + \Delta w(y,s)$.

Therefore, we derive from (76) an equation satisfied by h:

$$\frac{dh}{ds} = \left(pf_k\left(\frac{y(s)}{\sqrt{s}}\right)^{p-1} - \frac{1}{p-1}\right)h(s) + N(h) + r(y(s),s).$$

From (75) and homogeneity, we write $\forall s \in [s_0, s_*]$, $|N(h)| \leq C^*(K_0)|h|^2 \leq C^*(K_0)\frac{Be^{s-s_0}}{s_0}|h|$ and $|r(y(s),s)| \leq \frac{C_4}{s}$.

Therefore, if $g(s) = |h(s)|$, then $g(s)$ satisfies:

$$(79)\qquad \begin{cases} \forall s \in [s_0, s_*],\ g'(s) & \leq & \alpha(s)g(s) + \frac{C_4}{s}, \\ g(s_0) & \leq & \frac{(|a|+1)}{s_0} \end{cases}$$

with

$$(80)\qquad \alpha(s) = pf_k\left(\frac{y(s)}{\sqrt{s_1}}\right)^{p-1} - \frac{1}{p-1} + C^*(K_0)\frac{Be^{s-s_0}}{s_0}.$$

Using Gronwall's inequality, we write

$$\forall s \in [s_0, s_*],\ g(s) \leq I + II$$

where

$$(81)\qquad I = \exp\left(\int_{s_0}^{s}\alpha\right)g(s_0) \text{ and } II = C_4\int_{s_0}^{s}\frac{d\sigma}{\sigma}\exp\left(\int_{\sigma}^{s}\alpha\right).$$

We estimate in the following lemma $\exp\left(\int_\sigma^s\alpha\right)$ for $s_0 \leq \sigma \leq s \leq s_1$.

Lemma 3.1 *There exists $A_1(K_0) > 0$ such that $\forall A \geq A_1(K_0)$, $\exists s_{05}(K_0,A)$ such that $\forall s_0 \geq s_{05}(K_0,A)$, if $s_0 \leq \sigma \leq s \leq s_1$, then*

$$\exp\left(\int_\sigma^s\alpha\right) \leq \frac{3}{2}e^{s-\sigma}\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}.$$

We let the proof of this lemma to the end, and finish the proof of Proposition 3.2.

Now, we define $A_0(K_0) = A_1(K_0)$ and for each $A \geq A_0(K_0)$, $S_0(K_0,A) = \max(s_{04}(K_0,A), s_{05}(K_0,A))$. For $A \geq A_0(K_0)$ and $s_0 \geq S_0(K_0,A)$, we use (79) and lemma 3.1 to bound I and II (see (81)) for $s \in [s_0, s_*]$:

$I \leq (|a|+1)\frac{3}{2}\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\frac{e^{s-s_0}}{s_0}$ and

$II \leq C_4\frac{3}{2}\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\int_{s_0}^s\frac{d\sigma}{\sigma}e^{s-\sigma} \leq \frac{3}{2}C_4\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\frac{e^{s-s_0}}{s_0}$.

Hence, for $s = s_*$,

$|h(s_*)| = g(s_*) \leq I + II \leq \frac{3}{2}(|a|+1+C_4)\left[1 + \frac{(p-1)K_0^2}{4p}\right]^{\frac{p}{p-1}}\frac{e^{s_*-s_0}}{s_0} = \frac{B(K_0)}{2}\frac{e^{s_*-s_0}}{s_0}$ (see (75)).

This contradicts (78) and concludes the proof of Proposition 3.2, Proposition 3.1 and Theorem 2 also.

Proof of lemma 3.1:

From (80), (70) and (76), we have
$\alpha(s) = \frac{p}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}} - \frac{1}{p-1} + C^*(K_0)B\frac{e^{s-s_0}}{s_0}$ with

$$b(\omega) = b\omega^T A_k\omega \text{ and } b = \frac{(p-1)^2}{4p}. \tag{82}$$

Therefore,
$\int_\sigma^s \alpha(\tau)d\tau = \left[\tau + \ln\left(p-1+b(\omega)A^2\frac{e^{\tau-s_0}}{s_1}\right)^{-\frac{p}{p-1}} + C^*(K_0)\frac{Be^{\tau-s_0}}{s_0}\right]_\sigma^s$
$= s-\sigma+\ln\left(\frac{p-1+b(\omega)A^2\frac{e^{\sigma-s_0}}{s_1}}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}}\right)^{\frac{p}{p-1}} + C^*(K_0)\frac{B}{s_0}\left(e^{s-s_0}-e^{\sigma-s_0}\right)$. This implies that
$\exp\left(\int_\sigma^s\right) = e^{s-\sigma}\left(\frac{p-1+b(\omega)A^2\frac{e^{\sigma-s_0}}{s_1}}{p-1+b(\omega)A^2\frac{e^{s-s_0}}{s_1}}\right)^{\frac{p}{p-1}}\exp\left(C^*(K_0)\frac{B}{s_0}\left(e^{s-s_0}-e^{\sigma-s_0}\right)\right)$.
Since $\sigma \le s \le s_1$ and $Ae^{\frac{s_1-s_0}{2}} = K_0\sqrt{s_1}$, we have $Ae^{\frac{\sigma-s_0}{2}} \le K_0\sqrt{s_1}$ and $Ae^{\frac{s-s_0}{2}} \le K_0\sqrt{s_1}$. Therefore,
$\exp\left(\int_\sigma^s\right) \le e^{s-\sigma}\left[1+\frac{bK_0^2}{p-1}\right]^{\frac{p}{p-1}}\exp\left(\frac{C^*(K_0)K_0^2Bs_1}{A^2s_0}\right)$ (note that $b(\omega) \le b$, see (82)).

We now introduce $A_1(K_0) > 0$ such that for all $A \ge A_1(K_0)$,
$\exp\left(\frac{2C^*(K_0)K_0^2B}{A^2}\right) \le \frac{3}{2}$ and consider $A \ge A_1(K_0)$. Then, we introduce $s_{05}(K_0, A)$ such that for all $s_0 \ge s_{05}(K_0, A)$, $s_1 \le 2s_0$. Then,
for $s_0 \ge s_{05}(K_0, A)$, we have $\exp\left(\int_\sigma^s\right) \le \frac{3}{2}e^{s-\sigma}\left[1+\frac{bK_0^2}{p-1}\right]^{\frac{p}{p-1}}$, which concludes the proof of lemma 3.1. ■

Proof of Theorem 3:

The proof will follow from Proposition 3.1 and localization estimates. We consider $u(t)$ a solution of (1) which blows-up at time $T > 0$ at some point $x_0 \in \mathbb{R}^N$. By translation invariance, we take $x_0 = 0$. We assume 0 to be an isolated blow-up point of $u(t)$. Therefore, there exists $\epsilon_0 > 0$ such that 0 is the unique blow-up point $u(t)$ in $B(0, 2\epsilon_0)$.

We aim at proving the equivalence of the following behaviors for $u(t)$ near 0 and for w_0 (=w) defined in (2):

(A) $\forall R > 0, \sup_{|y|\le R}\left|w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\right| = o\left(\frac{1}{s}\right)$ as $s \to +\infty$,

(B) $\exists\epsilon_0 > 0$ such that $\|q_0(y,s)\|_{L^\infty(|y|\le\epsilon_0e^{s/2})} \to 0$ as $s \to +\infty$ where

$$q_0(y,s) = w(y,s) - f_0(\frac{y}{\sqrt{s}}) \tag{83}$$

and

$$f_0(z) = (p-1+\frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}, \tag{84}$$

(C) $\exists\epsilon_0 > 0$ such that if $|x| \le \epsilon_0$, then $u(x,t) \to u^*(x)$ as $t \to T$ and $u^*(x) \sim U(x)$ as $x \to 0$ where

$$U(x) = \left[\frac{8p|\log|x||}{(p-1)^2|x|^2}\right]^{\frac{1}{p-1}}. \tag{85}$$

For further purpose, we introduce a weaker version of (B) (which will be in fact equivalent):

(B') $\forall K_0 > 0$, $\|q_0(y,s)\|_{L^\infty(|y|\le K_0\sqrt{s})} \to 0$ as $s \to +\infty$.

The proof will be over if we prove the following implications:

$$(A) \Longrightarrow (B') \Longrightarrow (C) \Longrightarrow (B) \Longrightarrow (A).$$

We first prove some useful technical estimates. We then use them to prove the different implications.

Part I: Preliminary results for subcritical values of w ($w < \kappa$)
We crucially use the localization result proved in [16].

Lemma 3.2 *Assume that 0 is the only blow-up point of $u(t)$ in $B(0,2\epsilon_0)$ for some $\epsilon_0 > 0$. Consider (y_n, s_n) a sequence in $\mathbb{R}^N \times [-\log T, +\infty)$ satisfying $|y_n| \le \epsilon_0 e^{s_n/2}$ and suppose that $w(y_n,s_n) \to l \in (0,\kappa)$ and $s_n \to +\infty$ as $n \to +\infty$.*
If $x_n = y_n e^{-s_n/2}$ and $z_n = \frac{|y_n|}{\sqrt{s_n}}$, then:
i) $x_n \to 0$ as $n \to +\infty$,
ii) $\forall n \in \mathbb{N}$, $u(x_n,t) \to u^*(x_n)$ as $t \to T$ and
$$u^*(x_n) \sim \left[\frac{|x_n|^2}{2z_n^2\left|\log\frac{|x_n|}{z_n}\right|}\right]^{-\frac{1}{p-1}} \left(l^{1-p} - p + 1\right)^{-\frac{1}{p-1}} \text{ as } n \to +\infty.$$

Proof: We proceed by contradiction in order to prove that $x_n \to 0$ as $n \to +\infty$. If not, then we have $x_{n'} > \delta > 0$ for some subsequence $x_{n'}$. Since $u(t)$ does not blow-up for $\delta \le |x| \le \epsilon_0$, there exists $C(\delta) > 0$ such that if $t \in [\frac{T}{2}, T)$ and $\delta \le |x| \le \epsilon_0$, then $|u(x,t)| \le C(\delta)$. Therefore, (2) implies that $0 \le w(y_{n'}, s_{n'}) \le e^{-\frac{s_{n'}}{p-1}} C(\delta) \to 0$ as $n \to +\infty$, which contradicts the fact that $l > 0$. Thus, $x_n \to 0$ as $n \to +\infty$.
Let us find an equivalent of $u^*(x_n)$.
We define for each $(\xi,\tau) \in \mathbb{R}^N \times [0,1)$

$$\begin{aligned} v_n(\xi,\tau) &= e^{-\frac{s_n}{p-1}} u(x_n + \xi e^{-\frac{s_n}{2}}, T + (\tau-1)e^{-s_n}) \\ &= (1-\tau)^{\frac{1}{p-1}} w\left(\frac{y_n+\xi}{\sqrt{1-\tau}}, s_n - \log(1-\tau)\right). \end{aligned} \tag{86}$$

Then v_n satisfies: $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0,1)$

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + v_n^p.$$

According to (6), $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that

$$\begin{cases} \left|\frac{\partial v_n}{\partial \tau}(0,\tau) - v_n(0,\tau)^p\right| \le \epsilon v_n(0,\tau)^p + C_\epsilon e^{-\frac{ps_n}{p-1}}, \forall \tau \in [0,1), \\ v_n(0,0) \to l. \end{cases}$$

Let us define first $v(\tau)$ as the solution of

$$v'(\tau) - v(\tau)^p = 0,\ v(0) = l,$$

that is $v(\tau) = \left(l^{1-p} - \tau(p-1)\right)^{-\frac{1}{p-1}}$.
Thus, if we denote $v_n(0,\tau)$ by $y_n(\tau)$, we have: $\forall \epsilon > 0$, there exists $n_0(\epsilon)$ such that $\forall n \geq n_0(\epsilon)$

$$\begin{cases} |y_n'(\tau) - y_n(\tau)^p| & \leq & \epsilon(y_n^p + 1), \ \forall \tau \in [0,1) \\ |y_n(0) - l| & \leq & \epsilon. \end{cases}$$

Since $v(0) \leq v(\tau) \leq v(1) < +\infty$, it follows from continuity results on ordinary differential equations that $\sup\limits_{\tau \in [0,1)} |y_n(\tau) - v(\tau)| \leq \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.
In particular,
$\lim\limits_{\tau \to 1} v_n(0,\tau) = \lim\limits_{t \to T} y_n(t) \to v(1) = \left(l^{1-p} - p + 1\right)^{-\frac{1}{p-1}}$ as $n \to +\infty$.

From (86), we have $u^*(x_n) = \lim\limits_{t \to T} u(x_n,t) = \lim\limits_{\tau \to 1} e^{\frac{s_n}{p-1}} v_n(0,\tau)$. Therefore,

$$e^{-\frac{s_n}{p-1}} u^*(x_n) \sim \left(l^{1-p} - p + 1\right)^{-\frac{1}{p-1}} \text{ as } n \to +\infty. \tag{87}$$

Since

$$\frac{|x_n|}{z_n} = \sqrt{s_n e^{-s_n}}, \tag{88}$$

we get

$$s_n \sim 2\left|\log \frac{|x_n|}{z_n}\right| \tag{89}$$

and then

$$e^{\frac{s_n}{p-1}} \sim \left[\frac{|x_n|^2}{2z_n^2 \left|\log \frac{|x_n|}{z_n}\right|}\right]^{-\frac{1}{p-1}} \text{ as } n \to +\infty.$$

Combining this with (87) concludes the proof of lemma 3.2. ■

Corollary 3.1 *Under the assumptions of lemma 3.2, if* $\frac{u^*(x_n)}{U(x_n)} \to 1$ *as* $n \to +\infty$*, then* $w(y_n, s_n) - f_0(\frac{y_n}{\sqrt{s_n}}) \to 0$*, where* f_0 *is defined in (84).*

Proof: Let us show that $\frac{u^*(x_n)}{U(x_n)} \to 1$ implies that $f_0(\frac{y_n}{\sqrt{s_n}}) \to l$. From (85) and lemma 3.2, we get

$$\frac{l^{1-p} - p + 1}{z_n^2 \left|\log \frac{|x_n|}{z_n}\right|} \sim \frac{(p-1)^2}{4p\,|\log |x_n||} \text{ as } n \to +\infty. \tag{90}$$

We claim that

$$|\log |x_n|| \sim \frac{s_n}{2}. \tag{91}$$

Indeed, (90) and (89) imply that $z_n \sim \frac{C(p,l)}{\sqrt{s_n}} |\log |x_n||$. Using (88), we get from this $|x_n| e^{\frac{s_n}{2}} \sim C(p,l)\sqrt{|\log |x_n||}$ which gives $|\log |x_n|| \sim \frac{s_n}{2}$.
Combining (90), (89) and (91) gives

$$z_n^2 \to \frac{4p(l^{1-p} - p + 1)}{(p-1)^2},$$

that is $f_0(z_n) \to l$ as $n \to +\infty$ (by (84)). ■

Part II: Proof of Theorem 3

Now, we are able to prove the equivalence.

(A) $\Longrightarrow$ (B'):

One can easily see from (84) that $\forall R>0$,
$\sup_{|y|\le R}\left|f_0(\frac{y}{\sqrt{s}})-\left(\kappa-\frac{\kappa}{4ps}|y|^2\right)\right|=O\left(\frac{1}{s^2}\right)$.

By (A), it follows that $\forall R>0$, $\sup_{|y|\le R}\left|w(y,s)-f_0(\frac{y}{\sqrt{s}})-\frac{N\kappa}{2ps}\right|=o\left(\frac{1}{s}\right)$.

Proposition 3.1 applied with $k=0$ (and $A_k=I_N$) yields by (83): $\forall K_0>0$, $\|q_0(y,s)\|_{L^\infty(|y|\le K_0\sqrt{s})}\to 0$ as $s\to+\infty$, which is (B').

(B') $\Longrightarrow$ (C):

Since 0 is the only blow-up point of u in $B(0,2\epsilon_0)$, we can define $u^*(x)=\lim_{t\to T} u(x,t)$ for all $0<|x|\le\epsilon_0$. Let (x_n) be any sequence tending to zero in $\mathbb{R}^N$. Let us prove that $u^*(x_n)\sim U(x_n)$ as $n\to+\infty$ where U is defined in (85).

Fix $r_0>0$. If n is large enough, we can uniquely define $s_n\to+\infty$ and y_n by $r_0e^{-s_n/2}\sqrt{s_n}=|x_n|$ and $y_n=x_ne^{s_n/2}$. Since $z_n=\frac{|y_n|}{\sqrt{s_n}}=r_0>0$, it follows from (B') and (83) that $w(y_n,s_n)\to f_0(r_0)\in(0,\kappa)$. Applying lemma 3.2 yields

$$u^*(x_n)\sim\left[\frac{|x_n|^2}{2r_0^2\left|\log\frac{|x_n|}{r_0}\right|}\right]^{-\frac{1}{p-1}}\left(f_0(r_0)^{1-p}-p+1\right)^{-\frac{1}{p-1}}.$$

From (84), we have $f_0(r_0)^{1-p}-(p-1)=\frac{(p-1)^2}{4p}r_0^2$. Therefore,

$$u^*(x_n)\sim\left[\frac{(p-1)^2|x_n|^2}{8p\left|\log\frac{|x_n|}{r_0}\right|}\right]^{-\frac{1}{p-1}}$$

which is equivalent to $U(x_n)$ by (85).

(C) $\Longrightarrow$ (B):

We want to prove that $\|q_0(y,s)\|_{L^\infty(|y|\le\epsilon_0e^{s/2})}\to 0$ as $s\to+\infty$. We proceed by contradiction and assume the existence of $\epsilon>0$, $s_n\to+\infty$ and $|y_n|\le\epsilon_0e^{s_n/2}$ such that

(92) $$|q_0(y_n,s_n)|\ge\epsilon \text{ as } n\to+\infty.$$

We can assume that $w(y_n,s_n)\to l_1$ and $f_0\left(\frac{y_n}{\sqrt{s_n}}\right)\to l_2$. According to Theorem 1 and (84), $l_1,l_2\in[0,\kappa]$. Note that (92) yields

(93) $$|l_1-l_2|\ge\epsilon.$$

Let us consider three cases:

Case 1: $l_1\in(0,\kappa)$. From (93), $w(y_n,s_n)-f_0\left(\frac{y_n}{\sqrt{s_n}}\right)$ does not go to 0. Hence, from lemma 3.2 and corollary 3.1, $x_n=y_ne^{-s_n/2}\to 0$ and $\frac{u^*(x_n)}{U(x_n)}$ does not go to 1 as $n\to+\infty$. This contradicts (C).

Case 2: $l_1 = \kappa$. Note that (93) implies that $l_2 \le \kappa - \epsilon$. We claim the existence of y_n' such that

$$|y_n| \le |y_n'| \text{ and } w(y_n', s_n) = \frac{1}{2}\left(f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) + \kappa\right) \tag{94}$$

for large n. Indeed, w and f_0 are continuous, and we have

$$w(y_n, s_n) - \frac{1}{2}\left(f_0\left(\frac{y_n}{\sqrt{s_n}}\right) + \kappa\right) > 0$$

and

$$w(\frac{y_n}{|y_n|}\epsilon_0 e^{s_n/2}, s_n) - \frac{1}{2}\left(f_0\left(\frac{y_n}{|y_n|}\epsilon_0 \frac{e^{s_n/2}}{\sqrt{s_n}}\right) + \kappa\right) < 0$$

for large n (use (84) and write $w(\frac{y_n}{|y_n|}\epsilon_0 e^{s_n/2}, s_n) = e^{-\frac{s_n}{p-1}} u(\frac{y_n}{|y_n|}\epsilon_0, T - e^{-s_n})$ $\le C(\epsilon_0) e^{-\frac{s_n}{p-1}}$ since $u(t)$ does not blow-up for $|x| = \epsilon_0$).

We can assume that $w(y_n', s_n) \to l_1' \in [0, \kappa]$ (Theorem 1) and $f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \to l_2' \in [0, \kappa]$. Since f_0 is decreasing and $|y_n| \le |y_n'|$, we get $l_2' \le l_2 < \kappa$. Using (94), we get $l_1' = \frac{1}{2}(l_2' + \kappa) \in [\frac{\kappa}{2}, \kappa)$ and $|l_2' - l_1'| = \frac{1}{2}|\kappa - l_2'| > 0$.
Therefore, $w(y_n', s_n) - f_0\left(\frac{y_n'}{\sqrt{s_n}}\right)$ does not go to 0. Hence, from lemma 3.2 and corollary 3.1, $x_n' = y_n' e^{-s_n/2} \to 0$ and $\frac{u^*(x_n')}{U(x_n')}$ does not go to 1 as $n \to +\infty$. This contradicts (C).

Case 3: $l_1 = 0$. Note that (93) implies that $l_2 \ge \epsilon$. We claim the existence of y_n' such that

$$|y_n| \ge |y_n'| \text{ and } w(y_n', s_n) = \frac{1}{2} f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \tag{95}$$

for large n. Indeed, w and f_0 are continuous,

$$\lim_{n\to+\infty}\left[w(y_n, s_n) - \frac{1}{2} f_0\left(\frac{y_n}{\sqrt{s_n}}\right)\right] = -\frac{l_2}{2} \le -\frac{\epsilon}{2},$$

and

$$w(0, s_n) - \frac{1}{2} f_0(0) \to \frac{\kappa}{2}$$

($w(0, s_n) \to \kappa$ according to (7), since 0 is a blow-up point for $u(t)$). We can assume that $w(y_n', s_n) = \frac{1}{2} f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \to l_1' \le \frac{\kappa}{2}$ as $n \to +\infty$. Since $|y_n| \ge |y_n'|$, we have $f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \ge f_0\left(\frac{y_n}{\sqrt{s_n}}\right)$ and $2l_1' \ge l_2 \ge \epsilon > 0$. Therefore, $l_1' \in (0, \frac{\kappa}{2})$ and $w(y_n', s_n) - f_0\left(\frac{y_n'}{\sqrt{s_n}}\right) \to -l_1' < 0$. According to lemma 3.2 and corollary 3.1, $x_n' = y_n' e^{-s_n/2} \to 0$ and $\frac{u^*(x_n')}{U(x_n')}$ does not go to 1 as $n \to +\infty$. This contradicts (C).

(B) $\Longrightarrow$ (A):

According to (69), there exists $k \in \{0, 1, ..., N\}$ and a $N \times N$ orthonormal matrix Q such that

$$\forall R > 0, \quad \sup_{|y|\le R} |w(y, s) - f_k\left(\frac{y}{\sqrt{s}}\right) - \frac{a}{s}| = o\left(\frac{1}{s}\right) \tag{96}$$

where f_k and a are defined in (70).

Applying Proposition 3.1, we see that $\forall K_0 > 0$,
$\sup_{|z|\le K_0} \left|w(z\sqrt{s},s) - f_k(z)\right| \to 0$ as $s \to +\infty$.

Together with (B), this gives $f_k \equiv f_0$. Therefore, $k = 0$ and $a = \frac{N\kappa}{2p}$. Thus, (96) yields (A).

This concluses the proof of Theorem 3. ■

A Proof of lemma 2.4

i): - According to (32), $\forall i,j \in \{1,...,N\}$,
$w_{2,i,j}(s) = \int w(y,s)\left(\frac{1}{4}y_iy_j - \frac{1}{2}\delta_{i,j}\right)\rho(y)dy$.
Just remark that $\left(\frac{1}{4}y_iy_j - \frac{1}{2}\delta_{i,j}\right)\rho(y) = \frac{\partial^2\rho}{\partial y_i \partial y_j}$ and do two integrations by parts to get $w_2(s) = \int \nabla^2 w(y,s)\rho(y)dy$. The estimate for w_1 is similar.

ii): The estimates on w_1 and w_2 follow directly from *i)* since
$\|\nabla w(s_n)\|_{L^\infty} \le \frac{c_1}{\sqrt{s_n}}$ and $\|\nabla^2 w(s_n)\|_{L^\infty} \le \frac{A}{s_n}$.

- By (23), we write: $\forall y \in \mathbb{R}^N$,
$w(y,s_n) = w(0,s_n) + y.\nabla w(0,s_n) + \frac{1}{2}y^T\nabla^2 w(0,s_n)y + \phi(y,s_n)$ where

$$|\phi(y,s_n)| \le \frac{1}{6}\frac{A^{5/4}}{s_n^{3/2}}|y|^3. \tag{97}$$

According to (31), (30) and (28),

$$w_- = P_-(w) = P_-(\phi) = \phi_- \tag{98}$$

with notations similar to (31). From (31) and (97), we have
$|\phi_m(s_n)| \le C'(N)\frac{A^{5/4}}{s_n^{3/2}}$ for $m = 0,1,2$. Therefore, (31) yields $|\phi_-(y,s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1+|y|^3)$. Using (98), we get $|w_-(y,s_n)| \le C(N)\frac{A^{5/4}}{s_n^{3/2}}(1+|y|^3)$.

- Since $w(s)$ is well defined for all $s \ge -\log T$ and satisfies (3), lemma 2.3 implies that $w_0(s_n) \le \kappa$. Let us show that $w_0(s_n) \ge \delta_0 = \frac{c_2^3}{128C(N)^2A^{5/2}}$. We proceed by contradiction and assume that $w_0(s_n) < \delta_0$.
Consider $\hat{y}_n = \eta\frac{c_2\sqrt{s_n}}{4C(N)A^{5/4}}\varphi$ where φ is unitary and satisfies $\varphi^T w_2(s_n)\varphi = -\frac{c_2}{s_n}$ (use *i)* and Proposition 2.2), and $\eta \in \{-1,1\}$ is chosen so that
$w_1(s_n).\hat{y}_n \le 0$. Therefore, from (31) and the bounds on w_0, w_1, w_2 and w_-, we get:
$w(\hat{y}_n,s_n) = w_0(s_n) + w_1(s_n).\hat{y}_n + \left(\frac{1}{2}\hat{y}_n^T w_2(s_n)\hat{y}_n - trw_2(s_n)\right) + w_-(y,s_n)$
$\le \delta_0 + 0 - \frac{1}{2}\frac{c_2^2 s_n}{16C(N)^2A^{5/2}}\frac{c_2}{s_n} + \frac{C''(N)A}{s_n} + \frac{C(N)A^{5/4}}{s_n^{3/2}}\left(1 + \frac{c_2^3 s_n^{3/2}}{64C(N)^3A^{15/4}}\right)$
$= \delta_0 - \frac{c_2^3}{64C(N)^2A^{5/2}} + O\left(\frac{1}{s_n}\right) = -\delta_0 + O\left(\frac{1}{s_n}\right) < 0$ for s_n large enough. This contradicts the fact that w is nonnegative. Thus, $w_0(s_n) \ge \delta_0$.

iii): Since $M(y)$ defined in (32) is the matrix of eigenfunctions corresponding to the null eigenvalue of $\mathcal{L}$, we find the following equation if we multiply (27) by $M(y)\rho(y)$, integrate the expression over $\mathbb{R}^N$ and use (31):

$$w_2'(s_n) = -\frac{p}{p-1}w_2(s_n) + \int w(y,s_n)^p M(y)\rho(y)dy.$$

Thus, we focus on the computation of $\int w(y,s_n)^p M(y)\rho(y)dy$. Since $0<\delta_0\le w_0(s_n)\le\kappa$ and $0\le w(y,s_n)\le\kappa+1$, we can Taylor expand $w(y,s_n)$ around $w_0(s_n)$ until the third order and use (31) to write:
$\int w(y,s_n)^p M(y)\rho(y)dy = I+II+III+IV+V+VI$ where
$I=\int w_0(s_n)^p M(y)\rho(y)dy=0$,
$II=\int pw_0(s_n)^{p-1}V(y,s_n)M(y)\rho(y)dy$,
$III=\int \frac{p(p-1)}{2}w_0(s_n)^{p-2}V(y,s_n)^2M(y)\rho(y)dy$,
$IV=\int \frac{p(p-1)(p-2)}{6}w_0(s_n)^{p-3}V(y,s_n)^3M(y)\rho(y)dy$
$V=O\left(\int |V(y,s_n)|^4|M(y)|\rho(y)dy\right)$ and

$$V(y,s_n)=w_1(s_n).y+\left(\frac{1}{2}y^Tw_2(s_n)y-trw_2(s_n)\right)+w_-(y,s_n). \tag{99}$$

Using (99), the orthogonality (in $L^2_\rho(\mathbb{R}^N)$) of y and $M(y)$ on one hand, and $M(y)$ and $w_-(y,s_n)$ on the other, we write:
$II=pw_0(s_n)^{p-1}\int\left(\frac{1}{2}y^Tw_2(s_n)y-trw_2(s_n)\right)M(y)\rho(y)dy$
$=pw_0(s_n)^{p-1}w_2(s_n)$ by integration by parts.

From (99), we have:
$III=\frac{p(p-1)}{2}w_0(s_n)^{p-2}\int\Big[(w_1(s_n).y)^2+\left(\frac{1}{2}y^Tw_2(s_n)y-trw_2(s_n)\right)^2$
$+w_-(y,s_n)^2+2w_1(s_n).y\left(\frac{1}{2}y^Tw_2(s_n)y-trw_2(s_n)\right)+2w_1(s_n).yw_-(y,s_n)$
$+2\left(\frac{1}{2}y^Tw_2(s_n)y-trw_2(s_n)\right)w_-(y,s_n)\Big]M(y)\rho(y)dy.$
Using $ii)$, parity and simple but long calculations (based on integration by parts, (32) and (17)) that we omit, we find:
$IV=\frac{p(p-1)}{2}w_0(s_n)^{p-2}\Big[2w_1(s_n)\otimes w_1(s_n)+4w_2(s_n)^2+O\left(\frac{1}{s_n^3}\right)+0$
$+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)+O\left(\frac{1}{s_n^{5/2}}\right)\Big]$. Hence,
$III=p(p-1)w_0(s_n)^{p-2}\left[w_1(s_n)\otimes w_1(s_n)+2w_2(s_n)^2\right]+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$
$+O\left(\frac{1}{s_n^{5/2}}\right)$.

As for III, one can expand $V(y,s_n)^3$ and $V(y,s_n)^4$, and use $i)$ to get: $IV=O\left(\frac{1}{s_n^{5/2}}\right)+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$ and $V=O\left(\frac{1}{s_n^{5/2}}\right)+O\left(\frac{|w_1(s_n)|}{s_n^{3/2}}\right)$.

Gathering all the previous bounds on I, II, III, IV and V yields $iii)$. This concludes the proof of lemma 2.4. ■

B Proof of Proposition 2.3

In [16], the same result has been proved in the case of one fixed solution (Theorem 3). Hence, we should adopt here the same strategy as for the proof of Theorem 3 in [16]. In fact, we will focus only on points which are different from [16] (energy estimates and a compactness procedure), and summarize the other arguments. We give the proof in two steps. We first use a compactness procedure and then proceed by contradiction in a second step in order to conclude the proof.

Step 1: Compactness Procedure

We proceed by contradiction and assume that for some $\eta_0>0$ and for all $k\in\mathbb{N}$, there are $s_k^*\ge-\log T$, w_k solution of (3) defined for all $s\ge$

$-\log T$ and satisfying $w_k \in \hat{V}_A(s_k^*)$, $x_k \in \mathbb{R}^N$ and $t_k \in [0,t_k^*]$ such that $|\Delta u_k(x_k,t_k)| \geq \eta_0 u_k(x_k,t_k)^p + k$ where $t_k^* = T - e^{-s_k^*}$ and $u_k(x,t) = (T - t)^{\frac{1}{p-1}} w_k\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right)$.
Let us introduce $U_k(x,t) = u_k(x+x_k,t)$ and $W_k(y,s) = e^{-\frac{s}{p-1}} U_k(ye^{-\frac{s}{2}}, T - e^{-s})$. Therefore, U_k is a solution of (1), W_k is a solution of (3),

$$\forall s \in [-\log T, s_k^*],\ W_k(s) \in V_A(s), \tag{100}$$

$$\text{and } |\Delta U_k(0,t_k)| \geq \eta_0 U_k(0,t_k)^p + k \tag{101}$$

where $t_k \in [0,t_k^*]$.

We first notice that

$$t_k \to T \text{ as } k \to +\infty.$$

Indeed, if not, then $t_{k'} \leq T - \delta_0$ where $\delta_0 > 0$ for some subsequence $t_{k'}$. Therefore, (100) implies that $|\Delta U_k(0,t_{k'})| \leq C(T-\delta_0)$ for k' large enough, which contradicts (101).

From (101) and (100), we have
$U_k(0,t_k) \leq \left(\frac{\Delta U_k(0,t_k)}{\eta_0}\right)^{\frac{1}{p}} \leq \left(\frac{A}{\eta_0}\right)^{\frac{1}{p}} \frac{(T-t_k)^{-\frac{1}{p-1}}}{|\log(T-t_k)|^{\frac{1}{p}}}$. Therefore,

$$W_k(0,s_k) = (T-t_k)^{\frac{1}{p-1}} U_k(0,t_k) \to 0 \text{ as } k \to +\infty \tag{102}$$

where $s_k = -\log(T-t_k)$. From Definition 2.1, (100) and compactness procedure, we derive the existence of U solution of (1) in $C^2(\mathbb{R}^N \times [0,T))$ such that $U_k \to U$ as $k \to +\infty$ in $C^2(K)$ for all compact subset of $\mathbb{R}^N \times [0,T)$.

Step 2: Energy estimates on U

We claim that U blows-up at time T at the point $x = 0$.

Let us first introduce the following localized energy for u:

$$\begin{aligned} \mathcal{E}_{a,t}(u) &= t^{\frac{2}{p-1}-\frac{N}{2}+1} \int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right] \rho\left(\frac{x-a}{\sqrt{t}}\right)dx \\ &+ \frac{1}{2(p-1)} t^{\frac{2}{p-1}-\frac{N}{2}} \int |u(x)|^2 \rho\left(\frac{x-a}{\sqrt{t}}\right)dx \end{aligned} \tag{103}$$

where ρ is introduced in (17).
It was proved in [11] that if the energy is small at some point $a \in \mathbb{R}^N$, then u does not blow-up at a. More precisely,

Proposition B.1 (Giga-Kohn) *Let u be a solution of equation (1).*

i) If for all $x \in B(x_0,\delta)$, $\mathcal{E}_{x,T-t_0}(u(t_0)) \leq \sigma$, then $\forall x \in B(x_0,\frac{\delta}{2})$, $\forall t \in (\frac{t_0+T}{2},T)$, $|u(t,x)| \leq \eta(\sigma)(T-t)^{-\frac{1}{p-1}}$ where $\eta(\sigma) \leq c\sigma^\theta$, $\theta > 0$, and c and θ depend only on p.

ii) **(Merle)** *Assume in addition that $\forall x \in B(x_0,\delta)$, $|u(\frac{t_0+T}{2},x)| \leq M$. There exists $\sigma_0 = \sigma_0(p) > 0$ such that if $\sigma \leq \sigma_0$, then $\forall x \in B(x_0,\frac{\delta}{4})$, $\forall t \in (\frac{t_0+T}{2},T)$, $|u(t,x)| \leq M^*$ where M^* depends only on M, δ, T and t_0.*

Proof: see Proposition 3.5 and Theorem 2.1 in [11] (see also [15]). ∎

Suppose that $U(x,t)$ does not blow-up at $(x,t) = (0,T)$, then [11] shows that $\mathcal{E}_{0,T-t}(U(t)) \to 0$ as $t \to T$. Therefore, we choose $t_0 > T$ such that $\mathcal{E}_{0,T-t_0}(U(t_0)) \leq \frac{\sigma_0}{4}$ where σ_0 is introduced in Proposition B.1. From a continuity argument in x, there is $R_0 > 0$ such that if $|x| \leq R_0$, then $\mathcal{E}_{x,T-t_0}(U(t_0)) \leq \frac{\sigma_0}{2}$.
Since $U_k(t_0) \to U(t_0)$ as $k \to 0$ in $C^2(K)$ for all K compact subset and $\|U_k(t_0)\|_{W^{1,\infty}} \leq C(t_0)$ by (100), we have for all $|x| \leq R_0$, $\mathcal{E}_{x,T-t_0}(U_k(t_0)) \leq \sigma_0$ for k large enough. From (100), we have $\|U_k\left(\frac{t_0+T}{2}\right)\|_{L^\infty} \leq C(t_0)$ for k large enough.
Applying Proposition B.1, we get for k large enough: $\forall |x| \leq R_0$, $\forall t \in (\frac{t_0+T}{2}, T)$, $|U_k(x,t)| \leq M(t_0, R_0)$. By parabolic regularity (see lemma 2.10 and its proof for a sketch of the technique), we get

$$\forall t \in (\frac{3t_0+T}{4}, T), \ |\Delta U_k(0,t)| \leq M'(t_0)$$

for k large enough, which contradicts (101). Therefore, U blows-up at time T at $x = 0$.

Step 3: Conclusion of the proof

We now follow the same ideas as for the Theorem 3 in [16]. We claim the existence of $t'_k < t_k$ such that

$$t'_k \to T \text{ and } W_k(0, s'_k) = (T-t'_k)^{\frac{1}{p-1}} U_k(0, t'_k) = \kappa_0 \tag{104}$$

where $s'_k = -\log(T-t'_k)$, $\kappa_0 \in (0,\kappa)$ satisfies $\forall t > 0$, $\forall a \in \mathbb{R}^N$, $\mathcal{E}_{a,t}(\kappa_0 t^{-\frac{1}{p-1}}) = \frac{\kappa_0^2}{2(p-1)} - \frac{\kappa_0^{p+1}}{p+1} \leq \frac{\sigma_0}{2}$ and σ_0 is defined in Proposition B.1. Since U blows-up at $x = 0$, $U(0,t)(T-t)^{\frac{1}{p-1}} \to \kappa$ as $t \to T$ by [11]. Hence, if $\delta > 0$ is small enough, then $\delta^{\frac{1}{p-1}} U(0, T-\delta) \geq \frac{3\kappa+\kappa_0}{4}$. Since $U_k(0,T-\delta) \to U(0,T-\delta)$ as $k \to +\infty$, we get $\delta^{\frac{1}{p-1}} U_k(0,T-\delta) \geq \frac{\kappa+\kappa_0}{2}$ for k large enough.

By (102) and continuity arguments, we have the existence of $t'_{\delta,k} \in [T-\delta, t_k]$ such that $(T-t'_{\delta,k})^{\frac{1}{p-1}} U_k(0, t'_{\delta,k}) = \kappa_0$. The existence of t'_k follows then from a diagonal process.

Let us define for all $\xi \in \mathbb{R}^N$ and $\tau \in [0,1)$,

$$v_k(\xi,\tau) = (T-t'_k)^{\frac{1}{p-1}} U_k\left(\xi\sqrt{T-t'_k}, t'_k + \tau(T-t'_k)\right). \tag{105}$$

Then, v_k is a solution of (1), and $v_k(\xi, 0) = (T-t'_k)^{\frac{1}{p-1}} U_k(\xi\sqrt{T-t'_k}, t'_k) = W_k(\xi, s'_k)$, where $s'_k = -\log(T-t'_k) \leq s^*_k$. Since $t'_k + \frac{3}{4}(T-t'_k) \leq t_k \leq t^*_k$ (the second estimate is true by construction, and the first follows from (102), (104) and techniques similar to those in lemma 2.7), it follows from (100) and (104) that $v_k(0,0) = \kappa_0$ and $\forall \tau \in [0, \frac{3}{4}]$

$$\|\nabla v_k(\tau)\|_{L^\infty} \leq \frac{C(p)c_1}{\sqrt{|\log(T-t'_k)|}}, \ \|\nabla^2 v_k(\tau)\|_{L^\infty} \leq \frac{C(p)A}{|\log(T-t'_k)|}, \tag{106}$$

and for k large enough and for all $|\xi| \le 4|\log(T-t'_k)|^{1/4}$, $\mathcal{E}_{\xi,1}(v_k(0)) \le 2\mathcal{E}_{\xi,1}(\kappa_0)$ $\le \sigma_0$. Therefore, from Proposition B.1 (applied with $\delta = 1$ and using translation invariance), we have $\forall \tau \in [\frac{1}{2}, 1]$, $\forall |\xi| \le 2|\log(T-t'_k)|^{1/4}$, $|v_k(\xi,\tau)| \le M(p)$.

Now using arguments similar to those of lemma 2.8, we get

$$(107)\ \forall \tau \in [\frac{3}{4}, 1),\ \forall |\xi| \le |\log(T-t'_k)|^{1/4},\ |v_k| + |\nabla v_k| + |\nabla^2 v_k| \le M(p).$$

By arguments similar to those of lemma 2.9, we get from (106) and (107) for k large enough,

$$\sup_{\tau \in [0,1]} |\Delta_\xi v_k(0,\tau)| \to 0 \text{ as } k \to +\infty.$$

Therefore, since v_k is a solution of (1), we have

$$\forall \tau \in [0,1),\ v_k(0,\tau) \ge \frac{\kappa_0}{2}$$

for k large enough. Hence

$$\forall \tau \in [0,1),\ |\Delta_\xi v_k(0,\tau)| \le \frac{\eta_0}{2} v_k(0,\tau)^p \qquad (108)$$

for k large enough, and this yields a contradiction.
Indeed, taking $\tau_k = \frac{t_k - t'_k}{T - t'_k}$, we get from (108) and (105): $\forall k \ge k_0$,
$|\Delta U_k(0,t_k)| = (T-t'_k)^{-\frac{p}{p-1}} |\Delta_\xi v_k(0,\tau_k)|$
$\le \frac{\eta_0}{2}(T-t'_k)^{-\frac{p}{p-1}} v_k(0,\tau_k)^p = \frac{\eta_0}{2} U_k(0,t_k)^p$, which contradicts (101).

This concludes the proof of Proposition 2.3.

C Proof of lemma 2.11

Define $\chi_1(\xi) = \chi_0(\frac{\xi}{2B_1})$ where χ_0 is defined in (63). Then, $\forall \xi \in \mathbb{R}^N$,

$$|\nabla \chi_1(\xi)| \le \frac{C}{B_1} 1_{\{|\xi| \ge 2B_1\}} \text{ and } |\Delta \chi_1(\xi)| \le \frac{C}{{B_1}^2} 1_{\{|\xi| \ge 2B_1\}}. \qquad (109)$$

Let $Z(\xi,\tau) = \chi_1(\xi) e^{-\lambda t} z(\xi,\tau)$. Then, we have from (56): $\forall \xi \in \mathbb{R}^N$, $\forall \tau \in [0,\tau_*]$,

$$\left\{ \begin{array}{l} \frac{\partial Z}{\partial \tau} \le \Delta Z + \mu + z e^{-\lambda \tau} \Delta \chi_1 - 2 e^{-\lambda \tau} \nabla.(z \nabla \chi_1), \\ Z(\xi,0) \le z_0,\ z(\xi,\tau) \le B_2. \end{array} \right. \qquad (110)$$

We now take $|\xi| \le B_1$ and use an integral formulation of (110) to write $Z(\xi,\tau) \le I + II + III + IV$ where
$I = \left(e^{\tau\Delta} Z(0)\right)(\xi)$, $II = \int_0^\tau ds e^{(\tau-s)\Delta}\mu$, $III = \int_0^\tau ds e^{(\tau-s)\Delta} e^{-\lambda s} z(s) \Delta\chi_1$ and $IV = -2\int_0^\tau ds e^{(\tau-s)\Delta} e^{-\lambda s} \nabla.\left(z(s)\nabla\chi_1\right)$.

From the maximum principle and (110), we have $I \le z_0$ and $II \le \mu \int_0^\tau ds \le \mu$.

The treatment of III and IV is similar. However, handling IV is a bit more delicate.
By an integration by parts, we have:
$IV = -2\int_0^\tau ds e^{-\lambda s} \nabla e^{(\tau-s)\Delta} z(s) \Delta\chi_1$

$= -2\int_0^\tau ds e^{-\lambda s}\int dx \left(-\frac{(\xi-x)}{2(\tau-s)}\right)\frac{e^{-\frac{|\xi-x|^2}{4(\tau-s)}}}{(4\pi(\tau-s))^{N/2}}z(x,s)\Delta\chi_1(x).$

From (110) and (109), we obtain:

$IV \le \int_0^\tau ds\int_{\{|x|\ge 2B_1\}} dx \frac{|\xi-x|}{\tau-s}\frac{e^{-\frac{|\xi-x|^2}{4(\tau-s)}}}{(4\pi(\tau-s))^{N/2}}\frac{CB_2}{{B_1}^2}.$

Since $|\xi| \le B_1$, $|x| \ge 2B_1$ and $0 \le \tau - s \le 1$, we have $e^{-\frac{|\xi-x|^2}{4(\tau-s)}} = e^{-\frac{|\xi-x|^2}{8(\tau-s)}}e^{-\frac{|\xi-x|^2}{8(\tau-s)}}$ $\le e^{-\frac{|\xi-x|^2}{8(\tau-s)}}e^{-\frac{{B_1}^2}{8}}$. Therefore,

$IV \le \frac{CB_2}{{B_1}^2}e^{-\frac{{B_1}^2}{8}}\int_0^\tau \frac{ds}{\sqrt{\tau-s}}\int_{\{|x|\ge 2B_1\}} dx\frac{|\xi-x|}{\sqrt{\tau-s}}\frac{e^{\frac{-|\xi-x|^2}{8(\tau-s)}}}{(4\pi(\tau-s))^{N/2}}$

$\le \frac{CB_2}{{B_1}^2}e^{-\frac{{B_1}^2}{8}}\int_0^\tau \frac{ds}{\sqrt{\tau-s}}\int |X|e^{-|X|^2}dX \le CB_2 e^{-\frac{{B_1}^2}{4}}.$

Similarly, we obtain: $III \le CB_2 e^{-\frac{{B_1}^2}{4}}$.

Combining the bounds on I, II, III and IV, we get the conclusion of lemma 2.11. ■

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Résumé: On s'intéresse au phénomène d'explosion en temps fini dans les équations du type:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u \tag{1}$$

où $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $1 < p$, $(N-2)p < N+2$.

Dans une première direction, on construit pour (1) une solution u qui explose en temps fini $T > 0$ en un seul point d'explosion $x_0 \in \mathbb{R}^N$, et on décrit complètement le profil (ou comportement asymptotique) de u à l'explosion. Cette construction s'appuie sur la technique d'estimations à priori des solutions explosives de (1) qui permet une réduction en dimension finie du problème, et sur un lemme de type Brouwer. La méthode utilisée permet de dégager un résultat de *stabilité* du comportement de la solution construite par rapport à des perturbations dans les données initiales ou dans le terme non linéaire de réaction. De plus, la méthode se généralise à des équations vectorielles de type chaleur avec non-linéarité sans structure de gradient, ainsi qu'au traitement d'un problème de reconnexion d'un vortex avec la paroi en supra-conductivité.

Dans une seconde direction, on s'intéresse à l'équation suivante associée à (1):

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y.\nabla w - \frac{w}{p-1} + w^p, \tag{2}$$

et on démontre un Théorème de Liouville qui donne une classification des solutions de (2) globales en temps et en espace et uniformément bornées. On obtient également une propriété de localisation de l'équation (1) (si $u \geq 0$) qui permet de la comparer de façon précise à la solution de l'équation différentielle associée.

Enfin, on s'intéresse de nouveau à la notion de profil et on utilise les estimations qui découlent du Théorème de Liouville pour prouver un résultat d'équivalence de différentes notions de profils d'explosion ou de développement asymptotique de u au voisinage de x_0 point d'explosion, en variable x, $y = \frac{x-x_0}{\sqrt{T-t}}$ ou $z = \frac{x-x_0}{\sqrt{(T-t)|\log(T-t)|}}$.

Mots clés: équation de la chaleur, singularité, explosion en temps fini, extinction en temps fini, profil, développement asymptotique, équations vectorielles, supra-conductivité.

Abstract: We are interested in finite-time blow-up phenomena for heat equations of the type (1).

We first construct for (1) a solution u which blows-up in finite time T at only one blow-up point $x_0 \in \mathbb{R}^N$, and describe completely its blow-up profile (or asymptotic behavior). This construction is based on *a priori* estimates' technique which reduces the problem to a finite-dimensional one, and on a Brouwer type lemma. This method allows us to derive a stability result of the behavior of u with respect to initial data or perturbation of the nonlinearity. In addition, we generalize the method to the case of vector-valued equations with a non gradient nonlinearity, as well as a vortex reconnection with the boundary in super-conductivity.

In a second step, we consider the equation (2) derived form (1), and prove a Liouville Theorem which classifies all uniformly bounded globally (in space and time) defined solutions of (2). We then obtain a localization property of equation (1) (if $u \geq 0$) which allows a precise comparison with solutions of the associated ordinary differential equation.

In a third step, we use a consequence of the Liouville Theorem to prove the equivalence of different notions of blow-up profile or asymptotic behavior near a blow-up point x_0 of u, namely in variables x, $y = \frac{x-x_0}{\sqrt{T-t}}$ or $z = \frac{x-x_0}{\sqrt{(T-t)|\log(T-t)|}}$.

Key words: heat equation, singularity, finite time blow-up, finite time quenching, profile, asymptotic behavior, vector-valued equations, super-conductivity.