QUANTUM MONODROMY FOR A HOMOCLINIC ORBIT

by

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Abstract. — We present here some results from [**2**]. We show how to construct a quantum monodromy operator for a Schrödinger operator, in the case where the corresponding hamiltonian vector field possesses a homoclinic orbit.

1. Introduction

In this talk, we shall explore one aspect of the correspondence between classical and quantum mechanics. Let us recall that, if ξ denotes the classical momentum, the classical energy E of a particle moving in the potential $V(x)$ is given by $E = \xi^2 + V(x)$. The equation of motion for such a particle is obtained considering E as the hamiltonian: The particle moves along the integral curves of the hamiltonian vector field $H_E(x,\xi) = \partial_\xi E \partial_x - \partial_x E \partial_\xi$. The corresponding quantum operator is $P(x, hD) = -h^2\Delta + V(x)$, and can be obtained, at least formally, by replacing ξ by $\frac{h}{i}\partial_x$.

In the case of a confining potential V for example, the classical energy E could take any values in $[V_{min}, +\infty[$, where V_{min} is the minimum of V. In the quantum case however, the Schrödinger operator P has discrete spectrum, and the possible energies of the particle are the different eigenvalues of P. In a scattering situation, say when the potential vanishes at infinity, P has continuous spectrum in $\mathbb{R}^+$, but one can define resonances for P as complex eigenvalues of a non-selfajoint operator associated to P. The eigenvalues of P, or rather the corresponding eigenfunctions, are interpreted as stationary states, and the resonant states, corresponding to some resonances, are interpreted as temporary states (or metastable states), whose lifetime is the inverse of the imaginary part of the resonance.

It is common knowledge in quantum mechanics, that to closed trajectories of H_p should correspond eigenvalues or resonances. In their paper [**11**], J.Sjöstrand and M. Zworski have shown how to associate a quantization of the Poincaré map to such a closed orbit. They have described how this quantum monodromy operator can be used to obtain very precise information on the spectrum of P, as for example a semiclassical trace formula. In a recent paper [**10**], J. Sjöstrand has used this idea to compute the resonances associated to a hyperbolic orbit in $p^{-1}(E_0)$, in a fixed (independent of h) neighboorhood of E_0, for a 2-dimensional Schrödinger operator.

In [**2**], borrowing many ideas from B. Helffer and J. Sjöstrand in [**6**], we study the case where the hamiltonian vector field H_p associated with a Schrödinger operator P possesses a homoclinic orbit. In particular we want to prove some results about resonances for such an operator that we have conjectured in [**1**], and recover the results that two of us have proved in the 1 dimensional case (see [**4**]). One of the main difference with respect to the situation in [**11**], is that there is no Poincaré map in the homoclinic case, or otherwise stated, that the motion of a classical particle on such a trajectory is not periodic. However, thanks to tunneling, a quantum particle cannot be (micro-)localized only on one branch of the homoclinic orbit, and one can still define a quantum monodromy operator.

2. The example of Sjöstrand and Zworski

In order to explain what this monodromy operator is, and how useful it is for eigenvalue problems, we reproduce here the discussion by J. Sjöstrand and M. Zworski in the introduction of [**11**].

Let $Q = \frac{h}{i}\partial_x$ on $L^2(\mathcal{S}^1)$. We ask the question of the existence of the resolvent $(Q-z)^{-1}$ for $z \in \mathbb{C}$, or in other words, that of the existence and uniqueness of the solutions to

$$(Q-z)u = f, \tag{1}$$

for $f \in L^2(\mathcal{S}^1)$. First, we try to solve $(Q-z)u = 0$. For what concerns uniqueness, we are lead to define the operator R_+ by $R_+u = u(0)$, and consider the problem, for $v \in \mathbb{C}$,

$$\begin{cases} (P-z)u = 0 \\ R_+u = v \end{cases} \tag{2}$$

Now we want to examine the possible existence of a solution. First we forget about the periodicity requirement, and, for a given initial data $v \in \mathbb{C}$, we define two solutions: $I_+(z)[v] : x \mapsto e^{izx/h}v$ on $]-\epsilon, 2\pi - 2\epsilon[$, and $I_-(z)[v] : x \mapsto e^{izx/h}v$ on $]-2\pi + 2\epsilon, \epsilon[$. It is convenient to say that $I_+(z)[v]$ is the forward solution with initial data v, and that $I_-(z)[v]$ is the backward solution.

It is then very natural to define the operator $M(z,h) : \mathbb{C} \to \mathbb{C}$, which associates to a given initial data $v \in \mathbb{C}$ for the forward solution I_+, the initial data $\tilde{v}$ for I_- such that

$$I_+(z)[v](\pi) = I_-(z)[\tilde{v}](-\pi) \tag{3}$$

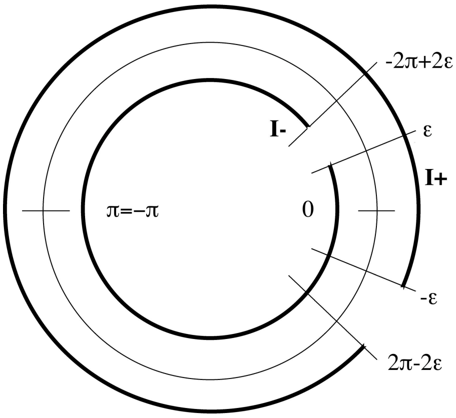

FIGURE 1. The forward and backward solutions.

In other words, the operator $M(z,h)$ is defined by

$$I_+(z)[v](\pi) = I_-(z)[M(z,h)v](-\pi). \tag{4}$$

This operator $M(z,h)$ is called the quantum mondromy operator.

It is obvious here that $M(z,h) = e^{2i\pi z/h}$. Of course, it is necessary for a 2π-periodic solution to exist with initial value v at $x = 0$, that $I_+(z)[v] = I_-(z)[v]$, and we have obtained the following description for the eigenvalues of Q:

$$\exists u \in L^2(\mathcal{S}^1) \text{ solution to (1)} \Longleftrightarrow \exists v \in \mathbb{C}, M(z,h)v = v \tag{5}$$

Going back to $(Q - z)$, we can get as easily (or see below) that

$$(Q - z) \text{ is not invertible} \Longleftrightarrow I - M(z,h) \text{ is not invertible.} \tag{6}$$

Thus we have obtained a Bohr-Sommerfeld type quantization rule:

$$z \in \sigma(Q) \Longleftrightarrow e^{2i\pi z/h} = 1, \tag{7}$$

which immediately leads to $\sigma(Q) = \mathbb{Z}$.

As a second, and perhaps more convincing application of this idea of monodromy operator, we explain how Sjöstrand and Zworski derive the usual Poisson formula. The point is that the equivalence (6) can be made more precise: $(P - z)^{-1}$ can be computed in terms of $(I - M(z,h))^{-1}$.

Indeed, let $\chi \in \mathcal{C}^\infty(\mathcal{S}^1)$ such that $\operatorname{supp}\chi \subset [-\epsilon, \pi+\epsilon]$, and $\chi \equiv 1$ on $[-\epsilon/2, \pi+\epsilon/2]$. If we put

$$E_+ = \chi I_+(z) + (1-\chi)I_-(z), \tag{8}$$

a simple computation gives

$$(P - z)E_+ = [P, \chi](I_+(z) - I_-(z)), \tag{9}$$

so that

$$(P-z)E_+ + [P,\chi]_\pi I_-(z)(I-M(z)) = 0, \tag{10}$$

Here we notice that the commutator $[P,\chi]$ can be written as a sum $[P,\chi]_0 + [P,\chi]_\pi$, where each term denotes the contribution to the commutator of a neighboorhood of 0 and π respectively. It is clear that only the last one contributes to (9), since $I_+(z)$ and $I_-(z)$ coincides at 0.

Now if we set $R_-(z) = [P,\chi]_\pi I_-(z)$, and

$$\mathcal{P}(z) = \begin{pmatrix} P-z & R_-(z) \\ R_+(z) & 0 \end{pmatrix}, \tag{11}$$

we see that, for any $v \in \mathbb{C}$, the Grusin problem

$$\mathcal{P}(z)\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{12}$$

has $(u = E_+ v, u_- = (I-M(z))v)$ for solution. In fact, one can show that the operator $\mathcal{P}(z)$ on $L^2(\mathcal{S}^1)\times\mathbb{C}$ is invertible, with inverse

$$\mathcal{E}(z) = \mathcal{P}(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}. \tag{13}$$

Every entry of $\mathcal{E}(z)$ is holomorphic with respect to z, and we have seen that $E_{-+}(z) = Id - M(z)$. Moreover we have the so-called Schur complement formula

$$(P-z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z). \tag{14}$$

Now let f be a function such that $\operatorname{supp}\hat{f} \subset [-2\pi N, 2\pi N]$ for some $N \in \mathbb{N}$. Let also $\Gamma = \Gamma_a \cup -\Gamma_{-a}$, where $\Gamma_\pm a = (\mathbb{R} \pm ia)$ $(a > 0)$, be an oriented contour in the complex plane around the real axis. Then we have

$$\begin{aligned} \sum_n f(n) = \operatorname{tr} f(P/h) &= \frac{1}{2i\pi}\int_\Gamma f(z/h)\operatorname{tr}(P-z)^{-1}dz \\ &= -\frac{1}{2i\pi}\int_\Gamma f(z/h)\operatorname{tr}(E_-E_+E_{-+}^{-1})dz \end{aligned} \tag{15}$$

Here we have used (14), removed the holomorphic parts which do not contribute to the contour integral, and used the cyclicity of the trace. We can also show easily that $E_-E_+ = \partial_z E_{-+}$, so that

$$\begin{aligned} \sum_n f(n) &= -\frac{1}{2i\pi}\int_\Gamma f(z/h)\operatorname{tr}(\partial_z E_{-+}E_{-+}^{-1})dz \\ &= \frac{1}{2i\pi}\int_{\Gamma_a} f(z/h)\operatorname{tr}\partial_z M(z)(I-M(z,h))^{-1}dz \\ &\quad - \frac{1}{2i\pi}\int_{\Gamma_{-a}} f(z/h)\operatorname{tr}\partial_z M(z)(I-M(z,h))^{-1}dz \end{aligned} \tag{16}$$

Then we write

$$(I - M(z,h))^{-1} = \sum_{k=0}^{N} M(z,h)^k + R_N(z,h), \tag{17}$$

and with a Paley-Wiener estimate for f, we can get rid of the integral of R_N on Γ_a letting $a \to +\infty$. A similar trick holds for the other integral, and we get, finally, the Poisson formula:

$$\sum_n f(n) = \frac{1}{2i\pi} \sum_{k=-N}^{N} \int_{\mathbb{R}} f(z/h) e^{2i\pi kz/h} \partial_z (e^{2i\pi z/h}) dz = \sum_{k=-N}^{N} \hat{f}(2\pi k). \tag{18}$$

3. Assumptions and some results

Let us detail the framework of [**2**]. We shall denote by P the Schrödinger operator on $L^2(\mathbb{R}^d)$ defined by

$$P(x, hD) = -\frac{h^2}{2}\Delta + V(x), \tag{19}$$

where V is a smooth function on $\mathbb{R}^d$. The trapped set $K(E)$ at energy E is defined by

$$K(E) = \{(x,\xi) \in p^{-1}(E), \exp tH_p(x,\xi) \not\to \infty \text{ as } t \to +\infty \text{ and as } t \to -\infty\}, \tag{20}$$

where $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical symbol of P.

We suppose that, for some energy level E_0, we have

(A1) : $p(0,0) = E_0$, and $(0,0)$ is an hyperbolic fixed point for H_p,
(A2) : $p(x,\xi) = E_0 \Rightarrow dp(x,\xi) \neq 0$ except for $(x,\xi) = (0,0)$.
(A3) : $K(E_0) = \{(0,0)\} \cup \gamma$, where γ is an integral curve for H_p, homoclinic for $(0,0)$:

$$\gamma \ :]-\infty, +\infty[\to T^*\mathbb{R}^d, \text{ with } \gamma(t) \to (0,0) \text{ both as } t \to -\infty \text{ and } t \to +\infty.$$

Since V has a local maximum at 0, we also have, in suitable coordinates,

$$V(x) = V(0) - \frac{1}{2}\sum_{j=1}^{d} \lambda_j^2 x_j^2 + O(x^3), \tag{21}$$

where we have ordered the eigenvalues of $\nabla^2 V(0)$ such that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d. \tag{22}$$

We make the following generic assumption on the λ_j's:

(A4) : We have $0 < \lambda_1 < \lambda_2$.

From now on we decide that $E_0 = V(0) = 0$. In the (x,ξ) coordinates, the linearized vector field F_p of H_p at $(0,0)$ is simply

$$F_p = d_{(0,0)}H_p = \begin{pmatrix} 0 & I \\ L^2 & 0 \end{pmatrix}, \tag{23}$$

where L is the $d \times d$ matrix defined as $L = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$. The eigenvalues of F_p are the λ_j's and the $-\lambda_j$'s.

Associated to the hyperbolic fixed point, we have therefore a natural decomposition of $T_{(0,0)}(T^*\mathbb{R}^d) = \mathbb{R}^{2d}$ in a direct sum of two linear subspaces Λ_+^0 and Λ_-^0, of dimension d, associated respectively to the positive and negative eigenvalues of F_p. These spaces $\Lambda_\pm^0$ are given by

$$\Lambda_\pm^0 : \xi_j = \pm\lambda_j x_j,\ j = 1 \dots d. \tag{24}$$

The stable/unstable manifold theorem gives us the existence of two Lagrangian manifolds Λ_+ and Λ_-, defined in a vicinity Ω of $(0,0)$, which are stable under the H_p flow and whose tangent space at (0,0) are precisely Λ_+^0 and Λ_-^0. In particular, we see that these manifolds can be written as

$$\Lambda_\pm : \xi = \nabla\phi_\pm(x), \tag{25}$$

for some smooth functions ϕ_+ and ϕ_-, which can be chosen so that

$$\phi_\pm(x) = \pm\sum_{j=1}^{d} \lambda_j x_j^2 + O(x^3). \tag{26}$$

We shall say that Λ_+ is the outgoing Lagrangian manifold, as Λ_- will be refered to as the incoming Lagrangian manifold associated to the hyperbolic fixed point. Indeed Λ_+ (resp. Λ_-) can be characterized as the set of points $(x,\xi) \in \Omega$ such that $\exp tH_p(x,\xi) \to (0,0)$ as $t \to -\infty$ (resp. as $t \to +\infty$).

In particular the points of the homoclinic curve γ which are in Ω belong either to Λ_+ or to Λ_-. Thus we can write $\gamma \cap \Omega = \gamma_+ \cup \gamma_-$ with $\gamma_\pm \subset \Lambda_\pm$ respectively. Because of the positions of $\Lambda_\pm^0$, this is of course a disjoint union (here we may have to shrink Ω). We can further suppose that, in Ω, the curve γ lies in the half-space $x_1 > 0$. Then, if $(x,\xi) \in \gamma_+$ (resp. $(x,\xi) \in \gamma_-$), we have $\xi_1 > 0$ (resp. $\xi_1 < 0$), thanks to (24) (See Figure 1). Notice that since we are working with a Schrödinger operator, and since we suppose that there is no other homoclinic curve in $p^{-1}(0)$, the homoclinic curve γ is symmetric with respect to $\xi = 0$.

Using the flow of H_p, we can extend $\Lambda_\pm$ as Lagrangian manifolds outside of Ω: we define $\Lambda_\pm^{\text{ext}}$ as

$$\Lambda_\pm^{\text{ext}} = \bigcup_{t \geq 0} \exp(\pm tH_p)(\Lambda_\pm) \tag{27}$$

The manifold Λ_+^{ext} (resp. Λ_-^{ext}) can be characterized as the set of points $(x,\xi) \in T^*\mathbb{R}^d$ such that $\exp tH_p(x,\xi) \to (0,0)$ as $t \to -\infty$ (resp. as $t \to +\infty$). Then it is clear

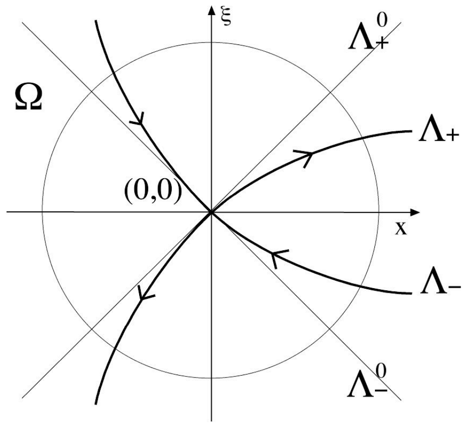

FIGURE 2. The geometry at the singular point

that the homoclinic curve γ belongs to both Λ_+^{ext} and Λ_-^{ext}. Here enters our fifth assumption:

(A5) : Λ_+^{ext} and Λ_-^{ext} intersect transversally along γ.

In this talk, we shall concentrate only on the singular part of the monodromy operator, that we define below.

We work in a domain of energies of the form $D(0, Ch) \subset \mathbb{C}$ for some $C > 0$ small enough. Let (x_0, ξ_0) be a point on $\gamma_- \subset \Omega$. For $hz \in D(0, Ch)$, we denote by $\mathcal{K}_{(x_0,\xi_0)}(z)$ the set of distributions microlocally defined near (x_0, ξ_0), such that $(P - hz)u = 0$. Let also $\mathcal{S}$ be the sphere on $\mathbb{R}^d$ with center at 0 and of radius $\|x_0\|$, or rather its lift in Λ_-. Of course $(x_0, \xi_0) \in \mathcal{S}$.

We consider the following problem

$$
(28) \qquad \begin{cases} (P - hz(h))u = 0 & \text{in } \Omega, \\ u = u_0 & \text{microlocally close to } (x_0, \xi_0), \\ u = 0 & \text{microlocally close to any } \ (x, \xi) \neq (x_0, \xi_0) \text{ in } \mathcal{S}. \end{cases}
$$

First, in the analytic category, provided V extends holomorphically in a strip around $\mathbb{R}^d$, we have the following uniqueness result :

Theorem 1. — *Let us denote by $\Gamma \subset \mathbb{C}$ the set given by*

$$
\Gamma = \{-i \sum_{j=1}^{d} \lambda_j (n_j + \frac{1}{2}), n_j \in \mathbb{N}\}
$$

If there is a $\delta > 0$ such that $d(z(h), \Gamma) \geq \delta$, the problem (28) has a at most one solution.

Roughly speaking, the exceptional set Γ corresponds to values of z for which there exist purely outgoing solutions, that is solutions which are microlocalized on Λ_+ only.

The definition of this set can be guessed by the results of J.Sjöstrand on barrier top resonances [**9**], and the proof of Theorem 1 is based on arguments from that paper (and also from [**3**]).

Notice that, since p is of principal type away from $(0,0)$, the microlocal kernel $\mathcal{K}_{(x_0,\xi_0)}(z)$ is isomorphic to $\mathcal{D}'(H_0)$, where $H_0 \subset \mathbb{R}^d$ is given by $x^1 = x_0^1$ (here, and from now on, we shall write points in $T^*\mathbb{R}^d$ as $(x,\xi) = (x^1, x', \xi^1, \xi')$ with x^1, ξ^1 in $\mathbb{R}$, x' and ξ' in $\mathbb{R}^{d-1}$). Indeed, there exists a h-Fourier integral operator (for short FIO) U, defined in a neighborhood of (x_0, ξ_0) such that, $U^{-1}PU = hD_{x_1}$ (see e.g. [**11**], Proposition 3.5). Then if we choose a hypersurface $\tilde{H}_0$ of $\mathbb{R}^d$ which is transverse to $\{x_1 = 0\}$, any element u_0 of $\mathcal{K}_{(x_0,\xi_0)}(z)$ can be associated to a unique initial data $\tilde{u}_0$ defined on $H_0 \sim \mathbb{R}^{d-1}$.

Now let $u_0 \in \mathcal{D}'(H_0)$, and suppose that there exists a function u solution to (28). We denote by v_0 the distribution defined microlocally close to $(x_0, -\xi_0) \in \gamma_+$ obtained by restriction of u, or rather the element of $\mathcal{D}'(H_0)$ it defines.

We call "singular part of the monodromy operator", and we denote by $\mathcal{I}_s(z)$, the operator on $\mathcal{D}'(H_0)$ which associates v_0 to u_0. Our aim here is to compute this operator $\mathcal{I}_s(z)$. Of course, this amounts to find a solution to (28). Following ideas from B. Helffer and J. Sjöstrand in [**6**], we have been able to show the

Theorem 2. — *There exist a neighborhood ω of 0 in $\mathbb{R}^d$, a neighborhood V of ξ_0' in $\mathbb{R}^{d-1}$, a phase function $\phi(t,x,\eta)$ and a symbol $a(t,x,\eta,h)$ defined on $[0,+\infty[\times\omega\times V$ such that*

$$u(x,h) = \iint_{T^*\mathbb{R}^{d-1}} \int_0^{+\infty} e^{i(\phi(t,x,\eta)-y\eta)/h} a(t,x,\eta,h) u_0(y) dt dy \frac{d\eta}{(2\pi h)^{d-1}}, \tag{29}$$

is a solution to (28).

Then, examining carefully the properties of the phase function ϕ, it appears that $\mathcal{I}_s(z)(u_0)$ is the "$t = +\infty$" part in the above integral, and we have obtained the

Theorem 3. — *Suppose $d(z(h), \Gamma) > \delta$ for some $\delta > 0$. Then the operator $\mathcal{I}_s(z)$ is well-defined, and it is a h-Fourier integral operator on $H_0 \sim \mathbb{R}^{d-1}$, associated to the canonical relation*

$$\mathcal{C} = \{(x', \nabla\tilde{\phi}_+(x'), y', \nabla\tilde{\phi}_-(y')), x', y' \in \mathbb{R}^{d-1}\},$$

where, for $x' \in \mathbb{R}^{d-1}$, $\tilde{\phi}_\pm(x') = \phi_\pm(x_0^1, x')$, and $\phi_\pm$ are the generating functions of $\Lambda_\pm$ given in (26).

In the two last sections, we explain briefly how to construct the phase function ϕ in Theorem 2, and we describe some of it properties leading to Theorem 3.

4. The phase function

Here we sketch the construction of the function ϕ in Theorem 2. We recall that H_0 is the hyperplane $x^1 = \epsilon$ in $\mathbb{R}^d$, where we denote $\epsilon = x_0^1$, since we want to emphasize the fact that we can work in a sufficiently small neighboorhood Ω of $(0,0)$ in $\mathbb{R}^d$. We also recall that (x_0, ξ_0) is the only point of γ_- above H_0.

Since γ_- is a simple characteristic for the operator p, by usual Hamilton-Jacobi theory we have first the

Proposition 4. — *For all $\eta \in \mathbb{R}^{d-1}$ close enough to ξ_0', there is a unique function $\psi_\eta : \mathbb{R}^d \to \mathbb{R}$, defined in a neighborhood ω_0 of x_0 such that*

$$\left\{\begin{array}{l} p(x, \nabla\psi_\eta(x)) = 0, \\ \psi_\eta(x) = x'.\eta,\ x \in H_0 \cap \omega_0, \\ \nabla\psi_\eta(x) \text{ is close to } \xi_0. \end{array}\right. \tag{30}$$

If we denote by Λ_ψ^η the corresponding Lagrangian manifold

$$\Lambda_\psi^\eta = \{(x,\xi) \in T^*\mathbb{R}^d, x \in \omega_0, \xi = \nabla\psi_\eta(x)\}, \tag{31}$$

we have the following

Proposition 5. — *The Lagrangian manifolds Λ_- and Λ_ψ^η intersect along an integral curve γ^η for H_p, and they intersect transversally. This curve is γ_- when $\eta = \xi_0'$.*

Again, one can see that, for η close enough to ξ_0' there is exactly one point $\rho(\eta) = (\rho_x(\eta), \rho_\xi(\eta))$ in γ^η above H_0. If we denote by Γ_0^η the set of level $\psi_\eta(\rho_x(\eta))$ for ψ_η

$$\Gamma_0^\eta = \{(x,\xi) \in \Lambda_\psi^\eta, \psi_\eta(x) = \psi_\eta(\rho_x(\eta))\}, \tag{32}$$

we can produce a Lagrangian manifold Λ_0^η, which contains Γ_0^η, such that Λ_0^η and Λ_ψ^η intersect transversally along Γ_0^η. Moreover, we can choose this Lagrangian manifold Λ_0^η such that it projects nicely on the x-space. Therefore there is a smooth function ϕ_0 such that, locally near $\rho(\eta)$, Λ_0^η is given by

$$\xi = \nabla_x\phi_0(x, \eta). \tag{33}$$

We denote by $\phi(t, x, \eta)$ the solution of the following eikonal equation

$$\left\{\begin{array}{l} \partial_t\phi(t,x,\eta) + p(x, \nabla_x\phi(t,x,\eta)) = 0, \\ \phi(0,x,\eta) = \phi_0(x,\eta), \end{array}\right. \tag{34}$$

and we consider the associated Lagrangian manifold Λ_t^η, given, for any $t \geq 0$ small enough, by

$$\xi = \nabla_x\phi(t,x,\eta). \tag{35}$$

Notice that we have of course

$$\Lambda_t^\eta = \exp(tH_p)(\Lambda_0^\eta). \tag{36}$$

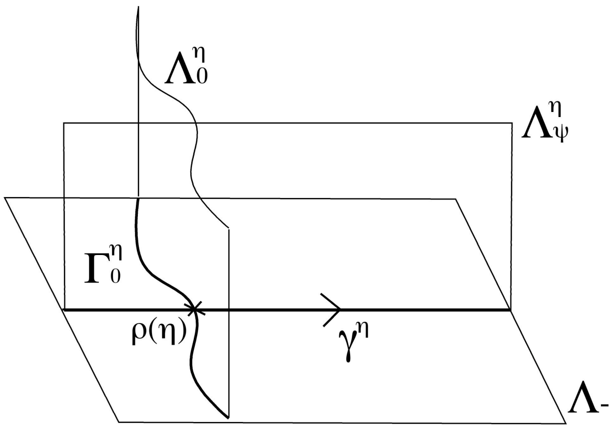

FIGURE 3. The Lagrangian manifolds.

It can be shown that, for any $t \geq 0$, Λ_t^η also projects itself nicely on the x-space in a neighborhood of $\rho_t(\eta) \in \Lambda_t^\eta$, where $\rho_t(\eta) = \exp(tH_p)\rho(\eta) \in \gamma^\eta$. If we define

$$\Gamma_t^\eta = \exp(tH_p)\Gamma_0^\eta, \tag{37}$$

we have the

Proposition 6. — *For each x close enough to γ^η, there is a unique time $t = t(x, \eta)$ such that $x \in \Pi_x \Gamma_t^\eta$. Moreover, it is the only critical point for the function $t \mapsto \phi(t, x, \eta)$, and it is a non-degenerate critical point.*

As a consequence of Proposition 6, we get in particular that, in ω_0 where both these functions are defined, we have

$$\nabla_x \psi_\eta(x) = \nabla_x(\phi(t(x, \eta), x)). \tag{38}$$

Therefore $x \mapsto \psi_\eta(x)$ and $x \mapsto \phi(t(x), x)$ differ by a constant. Until now however, the function ϕ_0 is also defined up to a constant, as well as ϕ, and we can choose ϕ_0 such that

$$\phi(t(x, \eta), x, \eta) = \psi_\eta(x), \tag{39}$$

and in particular, we get

$$\phi(t(x, \eta), x, \eta) = x'.\eta \tag{40}$$

for any $x \in H_0 \cap \omega_0$.

Now we go back to the representation formula (29), where ϕ is the function we have defined above. For $x \in H_0 \cap \omega_0$, a stationary phase expansion shows that, taking (40) into account, if u is the function defined by (29), we can choose the symbol a on H_0 such that $u(x, h) = u_0(x')$. Summing up, we have arranged so that the two last conditions in (28) are satisfied.

In order that the function u in (29) satisfies the first equation in (28), and since the phase ϕ is a solution to the eikonal equation (34), we only have to construct a symbol $a = \sum_j a_j h^j$ which satisfies the usual transport equations, for the initial data on H_0 we have fixed above. We shall not comment further on that point here.

5. Computation of $\mathcal{I}_s(z)$

We explain now how we compute the restriction v_0 to a microlocal neighborhood of $(x_0, -\xi_0) \in \gamma_+$, of the solution u defined in (29). Roughly, one could say that, as $u_0(x)$ is the contribution of the integral at the critical point $t(x)$, v_0 is the contribution of the integral at $t = +\infty$.

At this point, we need some general results from [**6**], here in a η-dependent setting.

5.1. Expandible symbols. — Let $(\mu_j)_{j\geq 0}$ be the strictly growing sequence of linear combinations over $\mathbb{N}$ of the λ_j's. For a function $w(t,x,\eta)$ defined on $\mathbb{R}^+ \times \omega \times V$, $\omega \subset \mathbb{R}^d$, $V \subset \mathbb{R}^{d-1}$, we shall write

$$w(t,x,\eta) = \tilde{\mathcal{O}}(e^{-\mu t}|x|^M) \tag{41}$$

when

$$\forall \epsilon > 0, w(t,x,\eta) = O(e^{-(\mu-\epsilon)t}|x|^M), \tag{42}$$

uniformly with respect to $\eta \in V$.

Definition 1. — *We say that $u : [0,+\infty[\times\omega \times V \to \mathbb{R}$, a smooth function, is expandible, if, for any $N \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$,*

$$\partial_t^k \partial_x^\alpha \left(u(t,x,\eta) - \sum_{j=1}^{N} u_j(t,x,\eta) e^{-\mu_j t} \right) = \tilde{\mathcal{O}}(e^{-\mu_{N+1} t}) \tag{43}$$

for a sequence of (u_j) smooth functions, which are polynomials in t. We shall write

$$u(t,x,\eta) \sim \sum_{j\geq 1} u_j(t,x,\eta) e^{-\mu_j t}$$

when (43) holds.

As the following result shows, this symbol class is the suitable one for our geometric setting at $(0,0)$.

Proposition 7. — *(see [**6**], Section 3)*
Let $\nu(t,x)$ be a time-dependent vector field. Suppose that there exists a matrix-valued map $x \mapsto A(x)$ from ω to $\mathcal{M}_d(\mathbb{R})$ such that

1. *$A(0) = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_d)$, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$.*
2. *$(t,x) \mapsto \nu(t,x) - A(x)x$ is a smooth real expandible matrix.*

Then, if $v(t,x)$ is expandible and vanishes at $x=0$, the solution $u(t,x)$ to the Cauchy problem

$$\begin{cases} \partial_t u + \nu(t,x)u = v, \ t \geq 0, x \in \omega, \\ u_{|t=0} = 0, \end{cases} \tag{44}$$

is expandible.

In particular, this result shows that the function $t \mapsto \gamma_-(t)$ is expandible:

$$\gamma_-(t) \sim \sum_{j\geq 1} \gamma_j(t) e^{-\mu_j t}. \tag{45}$$

Moreover, one can see that the function γ_1 is a constant vector from $\ker(d_{(0,0)}H_p + \lambda_1)$, and we have to make the generic assumption that

(A6) : $\gamma_1 \neq 0$.

5.2. Asymptotic behaviour of the phase. — We have defined the function ϕ as a solution to the eikonal equation (34). In particular, we can only assume that it is defined in a small t-interval. But for each η fixed, we are in the situation of [**6**], Section 2, and the results there apply to the present case. Summing up, one can prove that ϕ is well-defined for all $t > 0$, and that

Proposition 8. — *There exists a neighborhood V of ξ_0' in $\mathbb{R}^{d-1}$ such that $(t,x) \mapsto \phi(t,x,\eta)$ is expandible:*

$$\phi(t,x,\eta) - (\phi_+(x) + \tilde{\psi}(\eta)) \sim \sum_{j\geq 1} e^{-\mu_j t} \phi_j(t,x,\eta).$$

Here $\tilde{\psi}$ is a generating function for Λ_-, in the sense that, the projection of Λ_- onto T^*H_0 can be written as the set of $(\nabla\tilde{\psi}(\eta), \eta)$'s, with $\eta \in V$.

Writing

$$u(x,h) = \int \int_0^{+\infty} e^{i\phi(t,x,\eta)/h} a(t,x,\eta,h) \hat{u}_0(\eta) dt \frac{d\eta}{(2\pi h)^{d-1}},$$

and considering the $t = +\infty$ part of the integral, we obtain

$$\mathcal{I}_s(z) u_0(x) = e^{i\phi_+(x)/h} \int e^{-i\tilde{\psi}(\eta)/h} a_\infty(x,\eta,h) \hat{u}_0(\eta) dt \frac{d\eta}{(2\pi h)^{d-1}}, \tag{46}$$

and this is what we have summed up in Theorem 3.

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