Semiclassical Gevrey operators on exponentially weighted spaces of holomorphic functions

Michael HITRIK * Richard LASCAR † Johannes SJÖSTRAND‡
Maher ZERZERI §

In honor of Shmuel Agmon

Abstract: We provide a general overview of the recent works [18], [19] by the authors, devoted to continuity properties of semiclassical Gevrey pseudodifferential operators acting on a natural scale of exponentially weighted spaces of entire holomorphic functions.

Contents

1 Introduction and statement of results

The purpose of this brief survey paper is to give a broad non-technical account of some recent work by the authors [18], [19], dealing with semiclassical Gevrey pseudodifferential operators acting on exponentially weighted spaces of entire holomorphic functions on $\mathbf{C}^n$. Generally speaking, Gevrey microlocal analysis [26], [27], [14], occupying an

*Department of Mathematics, University of California, Los Angeles CA 90095-1555, USA, hitrik@math.ucla.edu

†JAD - UMR 7351, Université Côte d'Azur Parc Valrose 06108 Nice Cedex 02, France, richard.lascar@univ-cotedazur.fr

‡IMB, Université de Bourgogne 9, Av. A. Savary, BP 47870 FR-21078 Dijon, France and UMR 5584 CNRS, johannes.sjostrand@u-bourgogne.fr

§LAGA - UMR7539 CNRS, Université Sorbonne Paris-Nord, 99, avenue J.-B. Clément F-93430 Villetaneuse, France, zerzeri@math.univ-paris13.fr

intermediate position between analytic microlocal analysis [36], [20] and standard microlocal analysis in the C^∞–framework [11], [21], [9], becomes of particular interest whenever there is an essential difference between the methods and results in the analytic and C^∞ categories. Such a difference occurs, in particular, when considering the action of a semiclassical pseudodifferential operator on microlocally exponentially weighted spaces, depending on the precise regularity properties of the (smooth) symbol. To motivate the following discussion and to arrive at a natural definition of the exponentially weighted spaces in question, it will be convenient for us to start by recalling the approach to semiclassical microlocal analysis based on FBI-Bargmann transforms, in the simplest metaplectic setting [36], [41], [30], [20]. To this end, let $\phi(x,y)$ be a holomorphic quadratic form on $\mathbf{C}^n_x \times \mathbf{C}^n_y$ such that

$$\operatorname{Im} \phi''_{yy} > 0, \quad \det \phi''_{xy} \neq 0. \tag{1.1}$$

Associated to ϕ is the (metaplectic) FBI-Bargmann transformation $T : \mathcal{S}'(\mathbf{R}^n) \to \mathrm{Hol}(\mathbf{C}^n)$, given by

$$Tu(x;h) = Ch^{-3n/4} \int e^{i\phi(x,y)/h}\, u(y)\, dy, \quad x \in \mathbf{C}^n,\ 0 < h \leq 1. \tag{1.2}$$

Here the constant $C > 0$ is chosen suitably so that the map T is unitary,

$$T : L^2(\mathbf{R}^n) \to H_{\Phi_0}(\mathbf{C}^n) := \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0/h} L(dx)), \tag{1.3}$$

where

$$\Phi_0(x) = \sup_{y \in \mathbf{R}^n} \left(-\operatorname{Im} \phi(x,y)\right), \tag{1.4}$$

and $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$. Heuristically, we can regard the map T in (1.2) as a way of passing from the real to the complex domain, and viewing T as a Fourier integral operator with complex phase, we introduce the associated complex linear canonical transformation

$$\kappa_T : \mathbf{C}^{2n} \ni (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y)) \in \mathbf{C}^{2n}. \tag{1.5}$$

The canonical transformation κ_T maps the real phase space $\mathbf{R}^{2n}$ bijectively onto the I-Lagrangian R-symplectic linear subspace

$$\Lambda_{\Phi_0} := \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right),\, x \in \mathbf{C}^n \right\} \subset \mathbf{C}^{2n} = \mathbf{C}^n_x \times \mathbf{C}^n_\xi, \tag{1.6}$$

see [20], and it follows, in particular, that the quadratic form Φ_0 in (1.4) is strictly plurisubharmonic. Furthermore, when $a \in C^\infty_b(\mathbf{R}^{2n})$ in the sense that $\partial^\alpha a \in L^\infty(\mathbf{R}^{2n})$, for all $\alpha \in \mathbf{N}^{2n}$, we have the exact Egorov property [41], [20],

$$T \circ a^w(x, hD_x) = b^w(x, hD_x) \circ T. \tag{1.7}$$

Here $b \in C_b^\infty(\Lambda_{\Phi_0})$ is given by $b = a \circ \kappa_T^{-1}$ and $a^w(x, hD_x)$, $b^w(x, hD_x)$ are the semiclassical Weyl quantizations of a and b, acting on $L^2(\mathbf{R}^n)$ and $H_{\Phi_0}(\mathbf{C}^n)$, respectively.

Let us proceed next to recall some general ideas concerning the philosophy of exponential weights on phase space and exponentially weighted estimates on the FBI-Bargmann transform side, [36], [41], [30]. Our starting point here is the well known observation that using the FBI transform point of view, one can give a systematic characterization of the semiclassical wave front set in the C^∞, analytic, and Gevrey frameworks. Indeed, let $u(h)$ be a tempered family in $L^2(\mathbf{R}^n)$, so that $|| \, u(h) \, ||_{L^2(\mathbf{R}^n)} \leq \mathcal{O}(h^{-K})$, for some $K \geq 0$. Let $(y_0, \eta_0) \in T^*\mathbf{R}^n = \mathbf{R}^{2n}$, and let us set $x_0 = \pi_x(\kappa_T(y_0, \eta_0))$, where $\pi_x : \Lambda_{\Phi_0} \ni (x, \xi) \mapsto x \in \mathbf{C}^n_x$ is the projection map. We then have respectively, $(y_0, \eta_0) \notin \mathrm{WF}_h(u)$, $(y_0, \eta_0) \notin \mathrm{WF}_{a,h}(u)$, $(y_0, \eta_0) \notin \mathrm{WF}_{s,h}(u)$, for some $s > 1$, precisely when there exists an open neighborhood $V \subset \mathbf{C}^n$ of x_0, such that, respectively,

- for all $N \in \mathbf{N}$, we have

$$|Tu(x; h)| \leq \mathcal{O}_N(1) h^N e^{\Phi_0(x)/h}, \quad x \in V, \quad 0 < h \leq 1, \tag{1.8}$$

  see [28], [43],

- there exists $C > 0$ such that

$$|Tu(x; h)| \leq \mathcal{O}(1) e^{\frac{1}{h}\left(\Phi_0(x) - \frac{1}{C}\right)}, \quad x \in V, \quad 0 < h \leq 1, \tag{1.9}$$

  see [36], [30],

- there exists $C > 0$ such that

$$|Tu(x; h)| \leq \mathcal{O}(1) e^{\frac{\Phi_0(x)}{h}} \exp\left(-\frac{1}{C} h^{-\frac{1}{s}}\right), \quad x \in V, \quad 0 < h \leq 1, \tag{1.10}$$

  see [7], [42].

The exponential growth properties of $Tu(x; h)$ near x_0, as $h \to 0^+$, reflect therefore the microlocal regularity of the family $u(h)$ near (y_0, η_0), and the estimates (1.8), (1.9), (1.10) suggest that in order to study microlocal properties of solutions to a pseudodifferential equation of the form $a^w(x, hD_x)u = 0$, say, where the symbol a is C^∞, analytic, or Gevrey on $\mathbf{R}^{2n}$, it may be natural to proceed as follows: passing to the FBI transform side via (1.7), one may consider deformations Φ of the quadratic weight function Φ_0 in (1.4), letting our (conjugated) pseudodifferential operator $b^w(x, hD_x) = T \circ a^w(x, hD_x) \circ T^{-1}$ act on the new spaces $H_\Phi(\mathbf{C}^n)$ of holomorphic functions, obtained from the Bargmann space $H_{\Phi_0}(\mathbf{C}^n)$ in (1.3) by modifying the exponential weight. Associated to the new spaces are the new I-Lagrangian manifolds of the form $\Lambda_\Phi \subset \mathbf{C}^{2n}$, and the natural symbol associated to the operator $b^w(x, hD_x)$ acting on the space $H_\Phi(\mathbf{C}^n)$ turns out to be the restriction of b to Λ_Φ, roughly speaking, see [36], [40], [20]. In the analytic case, in particular, this idea has proven to be extremely fruitful in the works by the third-named author [36], [37], [41], and we refer to [8], [17] for analogous developments in the C^∞ framework. In the latter case, deformations of the weight allowed are fairly weak and should be $\mathcal{O}(h\,|\log h|)$–close to Φ_0, while much stronger perturbations of the standard weight, that are small but independent of h, become available in the analytic theory. Following [18], [19], our focus here is on the "intermediate" Gevrey case, where, as suggested by (1.10), admissible deformations should be $\mathcal{O}(h^{1-\frac{1}{s}})$–close to Φ_0. Here $s > 1$ is the Gevrey index.

We are now ready to describe the precise assumptions and state the main results established in [18], [19]. Let $s > 1$. The (global) Gevrey class $\mathcal{G}^s_b(\mathbf{R}^m)$ consists of all functions $u \in C^\infty(\mathbf{R}^m)$ such that there exist $A > 0$, $C > 0$ such that for all $\alpha \in \mathbf{N}^m$, we have

$$|\partial^\alpha u(x)| \leq AC^{|\alpha|}(\alpha!)^s, \quad x \in \mathbf{R}^m. \tag{1.11}$$

Identifying the space Λ_{Φ_0} in (1.6) linearly with $\mathbf{C}^n_x$, via the projection map $\pi_x : \Lambda_{\Phi_0} \ni (x,\xi) \mapsto x \in \mathbf{C}^n_x$, we may then also define the Gevrey space $\mathcal{G}^s_b(\Lambda_{\Phi_0})$. Given $a \in \mathcal{G}^s_b(\Lambda_{\Phi_0})$, for some $s > 1$, and $u \in \mathrm{Hol}(\mathbf{C}^n)$ such that $ue^{-\Phi_0/h}$ is rapidly decaying on $\mathbf{C}^n$, let us introduce the semiclassical Weyl quantization of a acting on u,

$$a^w_\Gamma(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta. \tag{1.12}$$

Here $0 < h \leq 1$ is the semiclassical parameter and $\Gamma(x) \subset \mathbf{C}^{2n}_{y,\theta}$ is the natural $2n$–dimensional contour of integration given by

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right), \quad y \in \mathbf{C}^n. \tag{1.13}$$

Let $\Phi_1 \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$, the space of C^1 functions on $\mathbf{C}^n$ with a globally Lipschitz gradient, be such that

$$||\, \nabla^k(\Phi_1 - \Phi_0)\, ||_{L^\infty(\mathbf{C}^n)} \leq \frac{1}{C} h^{1-\frac{1}{s}}, \quad k = 0, 1, 2, \tag{1.14}$$

for some $C > 0$ sufficiently large, depending on a. The following result has been established in [18].

Theorem 1.1 *Let $\omega = h^{1-\frac{1}{s}}$ and let us introduce the following $2n$–dimensional Lipschitz contours for $j = 0, 1$ and $x \in \mathbf{C}^n$,*

$$\Gamma^{\Phi_j}_\omega(x): \quad \theta = \frac{2}{i}\frac{\partial \Phi_j}{\partial x}\left(\frac{x+y}{2}\right) + if_\omega(x-y), \quad y \in \mathbf{C}^n, \tag{1.15}$$

where

$$f_\omega(z) = \begin{cases} \overline{z}, & |z| \leq \omega, \\ \omega\dfrac{\overline{z}}{|z|}, & |z| > \omega. \end{cases} \tag{1.16}$$

Let $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, for some $1 < s \leq 2$, and let $\widetilde{a} \in \mathcal{G}_b^s(\mathbf{C}^{2n})$ be an almost holomorphic extension of a off the maximally totally real subspace Λ_{Φ_0}, such that $\operatorname{supp} \widetilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbf{C}^{2n}}(0, C_0)$, *for some $C_0 > 0$. We have for $j = 0, 1$,*

$$a_\Gamma^w(x, hD_x) - \widetilde{a}^w_{\Gamma_\omega^{\Phi_j}}(x, hD_x) = \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\right) : \\ L^2(\mathbf{C}^n, e^{-2\Phi_j/h} L(dx)) \to L^2(\mathbf{C}^n, e^{-2\Phi_j/h} L(dx)), \quad (1.17)$$

where the realization

$$\widetilde{a}^w_{\Gamma_\omega^{\Phi_j}}(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_\omega^{\Phi_j}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \widetilde{a}\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta \qquad (1.18)$$

satisfies

$$\widetilde{a}^w_{\Gamma_\omega^{\Phi_j}}(x, hD_x) = \mathcal{O}(1) : H_{\Phi_j}(\mathbf{C}^n) \to L^2(\mathbf{C}^n, e^{-2\Phi_j/h} L(dx)). \qquad (1.19)$$

Here we have set, similarly to (1.3),

$$H_{\Phi_1}(\mathbf{C}^n) = \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_1/h} L(dx)).$$

Remark. It follows from Theorem 1.1 that the operator $a_\Gamma^w(x, hD_x)$ in (1.12) extends from $H_{\Phi_0}(\mathbf{C}^n) \cap H_{\Phi_1}(\mathbf{C}^n)$ to a uniformly bounded map,

$$a_\Gamma^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n), \qquad (1.20)$$

for $1 < s \leq 2$, but the conclusion in (1.17), (1.18), (1.19) is considerably more precise and allows us to regard the operator $\widetilde{a}^w_{\Gamma_\omega^{\Phi_1}}(x, hD_x)$ in (1.18) as the corresponding uniformly bounded realization, which agrees with $a_\Gamma^w(x, hD_x)$ modulo a remainder which is optimally small, see (1.17) and (1.10). We would also like to emphasize that it is precisely due to choice of the Lipschitz contour (1.15), (1.16) that we are able to obtain such accurate remainder estimates. Here the restriction to the range of Gevrey indices $1 < s \leq 2$ seems natural, and as discussed in [18, Section 3], the existence of a contour $\widetilde{\Gamma}(x)$, such that the properties (1.17), (1.19) both hold for $s > 2$, with $\Gamma_\omega^{\Phi_j}(x)$ replaced by $\widetilde{\Gamma}(x)$, seems unlikely.

Remark. We refer to [18, Section 3] for results analogous to Theorem 1.1 when $s > 2$. It turns out that in this range, one should accept remainder terms that are larger than the ones in (1.17), when obtaining uniformly bounded realizations of the Gevrey operator $a^w(x, hD_x)$ on the exponentially weighted spaces $H_{\Phi_j}(\mathbf{C}^n)$, $j = 0, 1$.

Remark. As is seen in the statement of Theorem 1.1, a crucial role in the work [18] is played by the existence of almost holomorphic extensions of a symbol $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, belonging to the same Gevrey class. This can be established by an application of a classical result of Carleson [5] on the universal moment problem. In fact, as observed in [18, Section 3], alternatively, it would have been sufficient in Theorem 1.1 to work with an extension $\widetilde{a} \in C_b^1(\mathbf{C}^{2n})$ of a such that

$$\left|\overline{\partial}\widetilde{a}(\rho)\right| \leq \mathcal{O}(1)\exp\left(-\frac{1}{\mathcal{O}(1)}\mathrm{dist}\left(\rho, \Lambda_{\Phi_0}\right)^{-\frac{1}{s-1}}\right), \quad \rho \in \mathbf{C}^{2n}. \tag{1.21}$$

Such an extension can be obtained more directly by means of Mather's method [31], see [18, Section 2].

Let us now turn the attention to the work [19], which is concerned with mapping properties of semiclassical operators of the form $a^w(x, hD_x)$, for $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, with the Gevrey index s belonging to the "complementary" range $s \geq 2$. The following is the main result of that work.

Theorem 1.2 *Let $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, for some $s \geq 2$, and let $\Phi_1 \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$ be such that*

$$||\,\nabla^k(\Phi_1 - \Phi_0)\,||_{L^\infty(\mathbf{C}^n)} \leq \frac{1}{C}h^{1-\frac{1}{s}}, \quad k = 0, 1, \tag{1.22}$$

where $C > 0$ is large enough, depending on a. Then the operator $a_\Gamma^w(x, hD_x)$ in (1.12) *extends from $H_{\Phi_0}(\mathbf{C}^n) \cap H_{\Phi_1}(\mathbf{C}^n)$ to a uniformly bounded map*

$$a_\Gamma^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n). \tag{1.23}$$

Remark. Let us point out that the conclusion of Theorem 1.2 follows also from the analysis developed in [18], apart from the inessential difference that (1.14) demands also that the Hessian of the perturbation $\Phi_1 - \Phi_0$ should be small, whereas no such condition is required in (1.22). The principal merit of the work [19] is to be found, in fact, in the method of proof of this result, which is quite direct and does not depend on the passage to a complex neighborhood of Λ_{Φ_0} by means of an almost holomorphic extension, relying instead on some basic ideas of the time frequency analysis [12]. It may also be interesting to notice that the arguments of [19] appear to work in the range of Gevrey indices $s \geq 2$ only, which is precisely the region where the almost holomorphic techniques of [18] do not lead to optimal remainder estimates.

We would like to finish the introduction by acknowledging that the study of Gevrey classes has a long and distinguished tradition in the linear PDE theory, in particular in connection with the analysis of hyperbolic operators with multiple characteristics, see [4], [6], [22], [23], [24], [25], [32], [33], [42]. Let us also mention some of the more recent developments concerning Gevrey regularity issues arising in the theory of dynamical systems, in the context of trace formulas for Anosov flows [14], as well as in the theory of Landau damping for the Boltzmann equation [3], [10]. To the best of our knowledge, the analysis of semiclassical Gevrey pseudodifferential operators has not yet been pursued systematically within the framework of H_Φ–spaces in the literature, with the very interesting paper [42] being a notable exception, and the works [18], [19] are meant to be the first steps in this direction.

It is a great pleasure for us to dedicate this paper to Professor Shmuel Agmon, and to acknowledge his influential fundamental work on exponentially weighted estimates for Schrödinger operators and Agmon-Lithner metrics [1]. It played a crucial role in the work by Bernard Helffer and the third-named author on multiple wells and tunneling [15], [16], and in the work by Barry Simon [35], see also [9, Chapter 6] for a systematic discussion and [29] for an early result on the exponential decay of eigenfunctions. As discussed above, in our works [18], [19], (moderately weak) exponential weights appear on the FBI-Bargmann transform side and we expect the techniques developed in the proofs of Theorem 1.1 and Theorem 1.2 above to be relevant when studying the propagation of Gevrey singularities for solutions of semiclassical Gevrey pseudodifferential equations, as well as for the functional calculus of (selfadjoint) Gevrey operators.

2 Some ideas of the proofs of Theorems 1.1 and 1.2

Referring to [18] and [19] for the complete details of the proofs, here we shall merely indicate the main steps along the way. When doing so, we shall also attempt to point out some of the possible directions for the future work.

As alluded to in the introduction, see (1.21), the starting point in the proof of Theorem 1.1 is the existence of an extension $\widetilde{a} \in C_b^\infty(\mathbf{C}^m)$ of $a \in \mathcal{G}_b^s(\mathbf{R}^m)$, for some $s > 1$, such that we have

$$\left|\overline{\partial}\widetilde{a}(z)\right| \leq \mathcal{O}(1) \exp\left(-\frac{1}{C}\left|\operatorname{Im} z\right|^{-\frac{1}{s-1}}\right), \quad \left|\operatorname{Im} z\right| \leq \mathcal{O}(1), \tag{2.1}$$

for some $C > 0$. Following the approach of [31], see also [9, Chapter 8] and [34], and assuming for simplicity that $a \in \mathcal{G}_b^s(\mathbf{R}^m)$ is compactly supported, we may set

$$\widetilde{a}(z) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \psi\left(|y|\,|\xi|^{1-\frac{1}{s}}\right) e^{i(x+iy)\cdot\xi}\widehat{a}(\xi)\, d\xi, \quad z = x+iy, \tag{2.2}$$

where $\psi \in C_0^\infty(\mathbf{R})$ is equal to 1 near the origin and $\widehat{a}(\xi) = \int e^{-ix\cdot\xi} a(x)\, dx$ is the Fourier transform of a. The bound (2.1) follows from (2.2) thanks to the following well known decay estimate for $\widehat{a}$,

$$|\widehat{a}(\xi)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{\mathcal{O}(1)}\,|\xi|^{\frac{1}{s}}\right), \quad \xi \in \mathbf{R}^m, \tag{2.3}$$

see [18, Section 2].

Once the existence of a suitable almost holomorphic extension of $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, satisfying (1.21), has been established, the proof of Theorem 1.1 proceeds by relying on essentially two basic ingredients: contour deformations to "good contours" adapted to the weights, in the spirit of [36], justified by an application of Stokes' formula, and Schur's lemma, to control the operator norms in various weighted L^2–spaces. Let us describe briefly the first step in the proof, deforming the contour $\Gamma(x)$ in (1.13) to $\Gamma_{\omega}^{\Phi_0}(x)$ in (1.15), (1.16), via the intermediate family of contours

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right) + itf_{\omega}(x-y), \quad y \in \mathbf{C}^n, \tag{2.4}$$

for $t \in [0,1]$. Letting also $G_{[0,1],\omega}(x) \subset \mathbf{C}^{2n}_{y,\theta}$ stand for the $(2n+1)$–dimensional contour given by (2.4), parametrized by $(t,y) \in [0,1]\times \mathbf{C}^n$, we can write for $u \in \mathrm{Hol}(\mathbf{C}^n)$ such that $e^{-\Phi_0/h}u$ is rapidly decaying, in view of Stokes' formula,

$$a^w_{\Gamma}(x, hD_x)u = \widetilde{a}^w_{\Gamma^{\Phi_0}_{\omega}}(x, hD_x)u + Ru. \tag{2.5}$$

Here $\widetilde{a}^w_{\Gamma^{\Phi_0}_{\omega}}(x, hD_x)$ is given in (1.18) and

$$Ru(x) = \frac{1}{(2\pi h)^n}\iiint_{G_{[0,1],\omega}(x)} e^{\frac{i}{h}(x-y)\cdot\theta}u(y)\,\overline{\partial}\left(\widetilde{a}\left(\frac{x+y}{2},\theta\right)\right)\wedge dy \wedge d\theta. \tag{2.6}$$

Writing

$$Ru(x) = \int r(x,y;h)u(y)\,L(dy),$$

we see that the absolute value of the effective kernel $e^{-\Phi_0(x)/h}r(x,y;h)e^{\Phi_0(y)/h}$ of the operator R does not exceed

$$\mathcal{O}(1)\,h^{-n}\sup_{t\in[0,1]}\begin{cases}\exp\left(-\frac{C}{h}(t|x-y|)^2 - C_1(t\,|x-y|)^{-\frac{1}{s-1}}\right), & |x-y|\leq \omega,\\ \\ \exp\left(-\frac{C}{h}\omega\, t\,|x-y| - C_1(t\,\omega)^{-\frac{1}{s-1}}\right), & |x-y| > \omega.\end{cases} \tag{2.7}$$

We would like to conclude that the quantity in (2.7) is

$$\mathcal{O}(h^{-n})\exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right),$$

uniformly, and some explicit computations show that this is indeed the case, provided that we choose the parameter $\omega = h^{1-\frac{1}{s}}$. An application of Schur's lemma implies then that

$$R = \mathcal{O}(1)\exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)) \to L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)), \tag{2.8}$$

and using Schur's lemma again we also obtain

$$\widetilde{a}^w_{\Gamma^{\Phi_0}_{\omega}}(x, hD_x) = \mathcal{O}(1)\max\left(1, h^{-n\left(1-\frac{2}{s}\right)}\right) : L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)) \to L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)). \tag{2.9}$$

In particular, this operator is $\mathcal{O}(1)$, provided that $1 < s \leq 2$.

Remark. As discussed in [18, Section 3], performing a deformation to a contour of the form

$$\widetilde{\Gamma}(x): \quad \theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right) + i\overline{(x-y)}, \quad y \in \mathbf{C}^n,$$

natural in the analytic theory [36], [41], [20], rather than to the contour $\Gamma_\omega^{\Phi_0}(x)$ in (1.15), (1.16), allows us only to conclude that

$$a_\Gamma^w(x, hD_x) - \widetilde{a}_\Gamma^w(x, hD_x) = \mathcal{O}(1)\exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{2s-1}}\right): \\ L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)) \to L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)), \quad (2.10)$$

giving a remainder estimate which is not precise enough. Heuristically speaking, when working with Gevrey symbols, we should stay therefore much closer to the real domain than in the analytic case. The price that we have to pay for working with contours such as $\Gamma_\omega^{\Phi_0}(x)$ in (1.15), (1.16) is that the corresponding realization $\widetilde{a}^w_{\Gamma_\omega^{\Phi_0}}(x, hD_x)$ in (1.18) is uniformly bounded on $L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx))$ in the range $1 < s \leq 2$ only.

When studying the action of the operator $a_\Gamma^w(x, hD_x)$ on the weighted space $H_{\Phi_1}(\mathbf{C}^n)$, with Φ_1 satisfying (1.14), we perform an additional contour deformation in (2.5), passing from $\Gamma_\omega^{\Phi_0}(x)$ to $\Gamma_\omega^{\Phi_1}(x)$, see (1.15) and (1.16). Applying Stokes' formula and Schur's lemma once more, we get

$$\widetilde{a}^w_{\Gamma_\omega^{\Phi_0}}(x, hD_x) - \widetilde{a}^w_{\Gamma_\omega^{\Phi_1}}(x, hD_x) = \mathcal{O}(1)\exp\left(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\right): \\ H_{\Phi_0}(\mathbf{C}^n) \to L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)). \quad (2.11)$$

Here the contour $\Gamma_\omega^{\Phi_1}(x)$ is adapted to the weight Φ_1 and yet another application of Schur's lemma gives therefore

$$\widetilde{a}^w_{\Gamma_\omega^{\Phi_1}}(x, hD_x) = \mathcal{O}(1)\max\left(1, h^{-n(1-\frac{2}{s})}\right): H_{\Phi_1}(\mathbf{C}^n) \to L^2(\mathbf{C}^n, e^{-2\Phi_1/h}L(dx)). \quad (2.12)$$

This completes a sketch of the proof of Theorem 1.1.

The paper [18] is concluded by the analysis of the composition $a^w(x, hD_x) \circ b^w(x, hD_x) = (a\#b)^w(x, hD_x)$, for $a, b \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, for some $s > 1$. Let σ be the complex symplectic $(2,0)$–form on $\mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n$. Using the classical formula for the Weyl symbol of the composition,

$$(a\#b)(X) = \frac{1}{(\pi h)^{2n}}\iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{-2i\sigma(Y,Z)/h}a(X+Y)b(X+Z)\,dY\,dZ, \quad (2.13)$$

see [43, Chapters 4, 13], of which we provide a partially new derivation in [18], we verify that the symbol $c = a\#b$ satisfies $c \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, making use of the method of contour deformations. In doing so, we also observe that the standard asymptotic expansion for c, as $h \to 0^+$, see [9, Chapter 7], does not lead to some sharp control of the remainders in the expansions. Here we encounter the familiar phenomenon of the loss of Gevrey smoothness in stationary phase expansions, see [26], [27], [34], and [14].

Remark. A closely related natural problem, which seems to be open, concerns establishing the following Wiener type result for the algebra of Gevrey pseudodifferential operators, and for simplicity we shall only state it in the real domain: assume that an operator of the form $a^w(x, hD_x)$, with $a \in \mathcal{G}_b^s(\mathbf{R}^{2n})$, for some $s > 1$, is invertible on $L^2(\mathbf{R}^n)$, for all $h > 0$ small enough. It follows from the Beals lemma [9, Chapter 8] that the inverse satisfies $(a^w(x, hD_x))^{-1} = b^w(x, hD_x)$, for some $b \in C_b^\infty(\mathbf{R}^{2n})$, and the problem is to show that $b \in \mathcal{G}_b^s(\mathbf{R}^{2n})$. More generally, establishing a Beals type characterization for the space of semiclassical Gevrey operators, while being of some independent interest, perhaps, may also have applications to the functional calculus of (selfadjoint) Gevrey operators, via the Dynkin-Helffer-Sjöstrand Cauchy formula, see [9, Chapter 8] and also [2], for applications to the functional calculus for several (non-selfadjoint non-commuting) operators.

Remark. The H_Φ–techniques developed in [18] are likely to have applications to the study of the propagation of Gevrey singularities for solutions of Gevrey pseudodifferential equations, starting with the case of Gevrey operators of real principal type as a "warm-up" problem. In this connection, let us emphasize that the work [42] by V. Sordoni also uses the techniques of exponentially weighted estimates on the FBI-Bargmann transform side, to study Gevrey singularities for microhyperbolic operators. In this work, Gevrey operators on the transform side are considered directly on the level of scalar products in exponentially weighted spaces, and we believe that the explicit description of the operators on the transform side, realized with the help of suitable good contours, obtained in [18], may also be valuable for the future analysis. A more long term project in this direction concerns developing a systematic FBI transform approach to microlocal analysis on Gevrey manifolds, see also [14] for recent results in this direction.

We shall finally sketch the principal ideas behind the work [19]. The starting point here is the following essentially well known Wiener type characterization of the Gevrey class $\mathcal{G}_b^s(\mathbf{R}^m)$, in the spirit of [38], [39]. Let $\varphi_0 \in \mathcal{G}_b^s(\mathbf{R}^m; \mathbf{R})$ be compactly supported and such that $||\,\varphi_0\,||_{L^2(\mathbf{R}^m)} = 1$. Given $a \in \mathcal{S}'(\mathbf{R}^m)$, we have $a \in \mathcal{G}_b^s(\mathbf{R}^m)$, for some $s > 1$, precisely when the following holds,

$$\sup_{t \in \mathbf{R}^m} |\mathcal{F}(\varphi_t a)(\xi)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{C}\,|\xi|^{1/s}\right), \quad \xi \in \mathbf{R}^m, \tag{2.14}$$

for some $C > 0$. Here $\varphi_t(x) = \varphi_0(x - t)$ and $\mathcal{F}$ is the Fourier transformation. As observed in [19], the function φ_0 in (2.14) can also be taken to be the L^2–normalized real Gaussian,

$$\varphi_0(x) = C_m e^{-|x|^2}, \quad C_m > 0, \tag{2.15}$$

and it is this choice which is made in [19]. Passing to the FBI transform side by means of a suitable complex linear canonical transformation of the form κ_T in (1.5), and letting $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, we may restate (2.14) as follows,

$$|\mathcal{F}(\chi_T a)(Y)| \leq \mathcal{O}(1) \exp\left(-\frac{1}{C_0}|Y|^{1/s}\right), \quad Y \in \Lambda_{\Phi_0}, \tag{2.16}$$

for some $C_0 > 0$, uniformly in $T \in \Lambda_{\Phi_0}$. Here $\chi_T(Y) = \chi_0(Y - T)$ and $\chi_0 = \varphi_0 \circ \kappa_T^{-1} \in \mathcal{S}(\Lambda_{\Phi_0})$, where φ_0 is defined in (2.15), with $m = 2n$. Using the Fourier inversion formula on the real symplectic space $(\Lambda_{\Phi_0}, \sigma|_{\Lambda_{\Phi_0}})$, we can write

$$M(h)a(X) = \frac{1}{(\pi h)^n} \iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} \chi_0\left(\frac{X-T}{h^{1/2}}\right) \mathcal{F}_h(\chi_T a)(Y)\, dY\, dT. \tag{2.17}$$

Here $M(h) \asymp h^n$ [I] and $\mathcal{F}_h$ is the semiclassical (symplectic) Fourier transformation on Λ_{Φ_0}. We would now like to pass to the Weyl quantizations in (2.17), and the crucial observation, due to [13] in the real setting, is that operator $\chi_0^w((x, hD_x)/h^{1/2})$ is a rank one orthogonal projection onto $\mathbf{C}v_0$, where $v_0 \in H_{\Phi_0}(\mathbf{C}^n)$ is a coherent state of the form

$$v_0(x) = Ch^{-n/2} e^{ig(x)/h}. \tag{2.18}$$

Here g is a holomorphic quadratic form on $\mathbf{C}^n$ such that $\Phi_0(x) + \operatorname{Im} g(x) \asymp |x|^2$, $x \in \mathbf{C}^n$. Combining this observation with (2.17) and some Weyl calculus, we obtain the following decomposition,

$$\begin{aligned} &M(h)a^w(x, hD_x)u \\ &= \frac{C}{h^n} \iint_{(\Lambda_{\Phi_0})^2} e^{\frac{i\sigma(Y,T)}{2h}} \mathcal{F}_h(\chi_{-\frac{Y+T}{2}} a)\left(\frac{Y-T}{2}\right) (u, e^{\frac{i\sigma((x,hD_x),T)}{h}} v_0)\, e^{\frac{i\sigma((x,hD_x),Y)}{h}} v_0\, dY\, dT, \end{aligned} \tag{2.19}$$

for some $C \neq 0$, which represents the operator $a^w(x, hD_x)$ as a superposition of rank one kernels. Here the magnetic translations $e^{i\sigma((x,hD_x),Y)/h}$, $Y \in \Lambda_{\Phi_0}$, are unitary as operators

$$e^{i\sigma((x,hD_x),Y)/h} : H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n),$$

and

$$e^{i\sigma((x,hD_x),Y)/h} : H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_2}(\mathbf{C}^n),$$

where $\Phi_2(x) = \Phi_0(x) + \Phi_1(x + y) - \Phi_0(x + y)$, $Y = (y, \eta) \in \Lambda_{\Phi_0}$, see [19, Section 2].

Remark. Assume that $a \in \mathcal{S}'(\Lambda_{\Phi_0})$ is such that $(Y, T) \mapsto \mathcal{F}(\chi_T a)(Y) \in L^1(\Lambda_{\Phi_0} \times \Lambda_{\Phi_0})$. It follows then from (2.19) that the operator $a^w(x, hD_x)$ is of trace class on $H_{\Phi_0}(\mathbf{C}^n)$,

[I] Here we write $X \asymp Y$ for $X, Y \in \mathbf{R}$ if X, Y have the same sign (or vanish) and $X = \mathcal{O}(Y)$ and $Y = \mathcal{O}(X)$.

which corresponds to a well known statement in the real setting, see [9, Chapter 9]. Assuming instead that

$$|\mathcal{F}(\chi_T a)(Y)| \leq U(Y), \quad Y \in \Lambda_{\Phi_0},$$

uniformly in $T \in \Lambda_{\Phi_0}$, for some $U \in L^1(\Lambda_{\Phi_0})$, we obtain, as a consequence of (2.19) combined with Schur's lemma, that the operator

$$a^w(x, hD_x) = \mathcal{O}(1) : H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n),$$

see [19, Section 4]. We recover therefore in a simple way the L^2–boundedness property for the Wiener algebra of pseudodifferential operators, established in [38], [39], see also [13].

The proof of Theorem 1.2 proceeds along the similar lines, combining the decomposition (2.19) with Schur's lemma, and making crucial use of the decay estimate (2.16). The restriction $s \geq 2$ appears at the very end of the argument, when estimating a contribution to the integral kernel of the form

$$K(x, y) = \frac{\mathcal{O}(1)}{h^n} e^{(f(y)-f(x))/h} e^{-|x-y|^2/Ch},$$

where $f = \Phi_1 - \Phi_0$ satisfying (1.14) with $k = 0, 1$. Indeed, using (1.22), we see that $K = \mathcal{O}(1) : L^2(\mathbf{C}^n) \to L^2(\mathbf{C}^n)$, provided that $s \geq 2$.

References

[1] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes, 29. Princeton University Press, 1982.

[2] M. Andersson and J. Sjöstrand, *Functional calculus for non-commuting operators with real spectra via an iterated Cauchy formula*, J. Funct. Anal. **210** (2004), 341–375.

[3] J. Bedrossian, N. Masmoudi, and C. Mouhot, *Landau damping: paraproducts and Gevrey regularity*, Ann. PDE **2** (2016), 71 pp.

[4] M. D. Bronšteĭn, *The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity*, Trudy Moskov. Mat. Obshch. **41** (1980), 83–99.

[5] L. Carleson, *On universal moment problems*, Math. Scand. **9** (1961), 197–206.

[6] M. Christ, *Intermediate optimal Gevrey exponents occur*, Comm. Partial Differential Equations **22** (1997), 359–379.

[7] J.-M. Delort, *F.B.I. transformation. Second microlocalization and semilinear caustics*, Springer–Verlag, Berlin, 1992.

[8] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudo-spectra of semiclassical (pseudo)differential operators*, Comm. Pure Appl. Math. **57** (2004), 384–415.

[9] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, Cambridge University Press, Cambridge, 1999.

[10] E. Grenier, T. Nguyen, and I. Rodnianski, *Landau damping for analytic and Gevrey data*, `https://arxiv.org/abs/2004.05979`.

[11] A. Grigis and J. Sjöstrand, *Microlocal analysis for differential operators*, Cambridge University Press, 1994.

[12] K. Gröchenig, *Foundations of time frequency analysis*, Birkhäuser, 2001.

[13] K. Gröchenig and C. Heil, *Modulation spaces and pseudodifferential operators*, Integr. Equ. Oper. Theory **34** (1999), 439–457.

[14] Y. Guedes Bonthonneau and M. Jézéquel, *FBI transform in Gevrey classes and Anosov flows*, `https://arxiv.org/abs/2001.03610`.

[15] B. Helffer and J. Sjöstrand, *Multiple wells in the semiclassical limit.* I., Comm. Partial Differential Equations **9** (1984), 337–408.

[16] B. Helffer and J. Sjöstrand, *Puits multiples en limite semi-classique.* II. *Interaction moléculaire. Symétries. Perturbation*, Ann. Inst. H. Poincaré Phys. Théor. **42** (1985), 127–212.

[17] F. Hérau, J. Sjöstrand, and C. Stolk, *Semiclassical analysis for the Kramers-Fokker-Planck equation*, Comm. PDE, **30** (2005), 689–760.

[18] M. Hitrik, R. Lascar, J. Sjöstrand, and M. Zerzeri, *Semiclassical Gevrey operators in the complex domain*, `https://arxiv.org/abs/2009.09125`.

[19] M. Hitrik, R. Lascar, J. Sjöstrand, and M. Zerzeri, *Semiclassical Gevrey operators and magnetic translations*, `https://arxiv.org/abs/2009.09128`.

[20] M. Hitrik and J. Sjöstrand, *Two minicourses on analytic microlocal analysis*, "Algebraic and Analytic Microlocal Analysis", Springer Proceedings in Mathematics and Statistics **269** (2018), 483–540.

[21] L. Hörmander, *The analysis of linear partial differential operators* (vol. I–IV), Springer, Berlin, 1985.

[22] V. Ivriĭ, *Well-posedness in Gevrey classes of the Cauchy problem for non-strict hyperbolic operators*, Mat. Sb. (N.S.) **138** (1975), 390–413.

[23] K. Kajitani and T. Nishitani, *The hyperbolic Cauchy problem*, Lecture Notes in Mathematics, 1505. Springer-Verlag, Berlin, 1991.

[24] K. Kajitani and S. Wakabayashi, *Microhyperbolic operators in Gevrey classes*, Publ. Res. Inst. Mat. Sci. **25** (1989), 169–221.

[25] H. Komatsu, *Linear hyperbolic equations with Gevrey coefficients*, J. Math. Pures Appl. **59** (1980), 145–185.

[26] B. Lascar, *Propagation des singularités Gevrey pour des opérateurs hyperboliques*, Amer. J. Math. **110** (1988), 413–449.

[27] B. Lascar and R. Lascar, FBI–*transforms in Gevrey classes*, J. Anal. Math. **72** (1997), 105–125.

[28] B. Lascar and J. Sjöstrand, *Équation de Schrödinger et propagation des singularités pour des opérateurs pseudo-différentiels à caractéristiques réelles de multiplicité variable*, Astérisque **95** (1982), 167–207.

[29] L. Lithner, *A theorem of the Phragmén-Lindelöf type for second-order elliptic operators*, Ark. Mat. **5** (1964), 281–285.

[30] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Universitext, Springer-Verlag, New York, 2002.

[31] J. N. Mather, *On Nirenberg's proof of Malgrange's preparation theorem*, Proc. Liverpool Singularities-Symp. I, Lecture Notes in Mathematics **192** (1971), 116–120.

[32] M. Petrini and V. Sordoni, *Propagation of singularities for hyperbolic operators with double characteristics*, Ann. Mat. Pura Appl. **163** (1993), 199–222.

[33] L. Rodino, *Linear partial differential operators in Gevrey spaces*, World Scientific, 1993.

[34] M. Rouleux, *Absence of resonances for semiclassical Schrödinger operators with Gevrey coefficients*, Hokkaido Math. J. **30** (2001), 475–517.

[35] B. Simon, *Semiclassical analysis of low lying eigenvalues. II. Tunneling*, Ann. of Math. **120** (1984), 89–118.

[36] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque **95** (1982).

[37] J. Sjöstrand, *Analytic wavefront sets and operators with multiple characteristics*, Hokkaido Math. J. **12** (1983), 392–433.

[38] J. Sjöstrand, *An algebra of pseudodifferential operators*, Math. Res. Lett. **1** (1994), 185–192.

[39] J. Sjöstrand, *Wiener type algebras of pseudodifferential operators*, Séminaire sur les Équations aux Dérivées Partielles, 1994–1995, Exp. No. IV, 21 pp., École Polytech., Palaiseau, 1995.

[40] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60** (1990), 1–57.

[41] J. Sjöstrand, *Function spaces associated to global I-Lagrangian manifolds*, Structure of solutions of differential equations, Katata/Kyoto, 1995, World Scientific, 1996.

[42] V. Sordoni, *On Gevrey singularities of microhyperbolic operators*, J. Anal. Math. **121** (2013), 383–399.

[43] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics 138, AMS, 2012.